E11.4 A system described by the matrix equations

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 2 \end{bmatrix} x$$

Determine whether the system is controllable and observable

Answer:

The controllability matrix is

$P_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$, and $Det(P_c) = -1 \neq 0$, therefore, the system is controllable

The observability matrix is

$P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -6 \end{bmatrix}$, and $Det(P_o) = 0$, therefore, the system is unobservable

E11.9 consider the second-order system

$$\dot{x} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + [0]u$$

For what value of k_1 and k_2 is the system completely controllable?

Answer:

$P_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} k_1 & k_1 - k_2 \\ k_2 & -k_1 + k_2 \end{bmatrix}$, and $Det(P_c) = -k_1^2 + k_2^2 \neq 0$

So, the condition for complete controllable is $k_1^2 \neq k_2^2$

P11.10 The dynamics of a rocket are represented by

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

and state variable feedback is used, where $u=2x_1+x_2$ Determine the roots of the characteristic equation of this system and the response of the system when the initial conditions are $x_1(0) = 1$ and $x_2(0) = 0$.
answer:

$$\dot{x} = (A - BK)x = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} x$$

The characteristic equation is

$$Det(\lambda I - (A - BK)) = s(s+2) + 1 = 0$$

The roots are $s_{1,2}$=-1,
The time response of the system is

$$x(t) = \phi(t)x(0) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} x(0) = \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix}$$

$$\begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = L^{-1}\begin{bmatrix} \dfrac{s}{s^2+2s+1} \\ \dfrac{1}{s^2+2s+1} \end{bmatrix} = \begin{bmatrix} (1-t)e^{-t} \\ te^{-t} \end{bmatrix}$$

P11.16 a system represented by

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x + [0]u$$

We want to place the closed-loop poles at $s = -2 \pm j2$ Determine the required state variable feedback using Ackermann's formula. Assume that the complete state vector is available for feedback.
Answer:

Let $u = -Kx$
Then Ackermann's formula is $K = \begin{bmatrix} 0, & 0, & \cdots, & 1 \end{bmatrix} P_c^{-1} q(A)$, where q(A) is the desired characteristic polynomial, which in this case is

$$q(s) = s^2 + 4s + 8$$

$$P_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad P_c^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$q(A) = A^2 + 4A + 8I = \begin{bmatrix} 4 & 0 \\ 2 & 8 \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & 8 \end{bmatrix}$$

P11.18 A system has transfer function $T(s) = \dfrac{1}{(s+1)^2}$

(a) Find a matrix differential equation to represent this system

(b) Select a state variable feedback using u(t), and select the feedback gain so that the repeated roots are at $s = -\sqrt{2}$, where $y(t)=x_1(t)$

Answer:

A matrix differential equation representation is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$Let \quad u(t) = -k_1 x_1 - k_2 x_2$$

Then, the close-loop characteristic equation is

$$q(s) = s^2 + (2 + k_2)s + 1 + k_1 = 0$$

We desire the characteristic equation

$$s^2 + 2\sqrt{2}s + 2 = 0$$

We can obtain $k_1 = 1, \quad k_2 = 2\sqrt{2} - 2 = 0.828$

AP11.13 Consider the system represented in state variable form

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$where$$

$$A = \begin{bmatrix} 1 & 2 \\ -5 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} -4 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 6 & -4 \end{bmatrix}, \quad and \quad D = [0]$$

Verify that the system is observable and controllable. If so, design a full-state feedback law and an observer by placing the closed-loop system poles at $s_{1,2} = -1 \pm j$ and the observer poles at $s_{1,2} = -10$

Answer:

The controllability matrix is

$$P_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 1 & 10 \end{bmatrix}$$

And the observability matrix is

$$P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 26 & 52 \end{bmatrix}$$

Computing the determinants yields

$$\det P_c = -38 \neq 0 \quad and \quad \det P_o = 416 \neq 0,$$

Hence the system is controllable and observable. The controller gain matrix

$$K = \begin{bmatrix} 3.55 & 7.21 \end{bmatrix}$$

$$L = q(A)P_o^{-1}\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T$$

The observer gain matrix is

$$L = \begin{bmatrix} 1.38 \\ -0.67 \end{bmatrix}$$