E11.4 A system described by the matrix equations

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y = \begin{bmatrix} 0 & 2 \end{bmatrix} x
\]

Determine whether the system is controllable and observable

Answer:

The controllability matrix is

\[
P_c = [B \ AB] = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}, \quad \text{and} \quad \text{Det}(P_c) = -1 \neq 0, \text{ therefore, the system is controllable}
\]

The observability matrix is

\[
P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -6 \end{bmatrix}, \quad \text{and} \quad \text{Det}(P_o) = 0, \text{ therefore, the system is unobservable}
\]

E11.9 Consider the second-order system

\[
\dot{x} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} u \\
y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u
\]

For what value of \(k_1\) and \(k_2\) is the system completely controllable?

Answer:

\[
P_c = [B \ AB] = \begin{bmatrix} k_1 & k_1 - k_2 \\ k_2 & -k_1 + k_2 \end{bmatrix}, \quad \text{and} \quad \text{Det}(P_c) = -k_1^2 + k_2^2 \neq 0
\]

So, the condition for complete controllable is \(k_1^2 \neq k_2^2\)

P11.10 The dynamics of a rocket are represented by

\[
\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
y = \begin{bmatrix} 0 & 1 \end{bmatrix} x
\]
and state variable feedback is used, where \( u=2x_1+x_2 \) Determine the roots of the characteristic equation of this system and the response of the system when the initial conditions are \( x_1(0) = 1 \) and \( x_2(0) = 0 \).

**Answer:**

\[
\dot{x} = (A - BK)x = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} x
\]

The characteristic equation is

\[
Det(\lambda I - (A - BK)) = s(s + 2) + 1 = 0
\]

The roots are \( s_{1,2} = -1 \),

The time response of the system is

\[
x(t) = \phi(t)x(0) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} x(0) = \begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix}
\]

\[
\begin{bmatrix} \phi_{11} \\ \phi_{21} \end{bmatrix} = L^{-1} \begin{bmatrix} s \\ s^2 + 2s + 1 \\ 1 \\ s^2 + 2s + 1 \end{bmatrix} = \begin{bmatrix} (1 - t)e^{-t} \\ te^{-t} \end{bmatrix}
\]

**P11.16** a system represented by

\[
\dot{x} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u
\]

\[y = \begin{bmatrix} 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \end{bmatrix} u\]

We want to place the closed-loop poles at \( s = -2 \pm j2 \) Determine the required state variable feedback using Ackermann's formula. Assume that the complete state vector is available for feedback.

**Answer:**

Let \( u = -Kx \)

Then Ackermann’s formula is \( K = [0, 0, \ldots, 1]P_c^{-1}q(A) \), where \( q(A) \) is the desired characteristic polynomial, which in this case is

\[q(s) = s^2 + 4s + 8\]

\[
P_c = [B \quad AB] = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad P_c^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}
\]

\[q(A) = A^2 + 4A + 8I = \begin{bmatrix} 4 & 0 \\ 2 & 8 \end{bmatrix}\]

\[K = [2 \quad 8]\]

**P11.18** A system has transfer function \( T(s) = \frac{1}{(s + 1)^2} \)

(a) Find a matrix differential equation to represent this system
(b) Select a state variable feedback using \( u(t) \), and select the feedback gain so that the repeated roots are at \( s = -\sqrt{2} \), where \( y(t) = x_1(t) \)

Answer:

A matrix differential equation representation is

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y = \begin{bmatrix} 1 & 0 \end{bmatrix} x
\]

Let \( u(t) = -k_1 x_1 - k_2 x_2 \)

Then, the close-loop characteristic equation is

\[
q(s) = s^2 + (2 + k_2)s + 1 + k_1 = 0
\]

We desire the characteristic equation

\[
s^2 + 2\sqrt{2}s + 2 = 0
\]

We can obtain \( k_1 = 1, \quad k_2 = 2\sqrt{2} - 2 = 0.828 \)

AP11.13 Consider the system represented in state variable form

\[
\dot{x} = Ax + Bu \\
y = Cx + Du
\]

where

\[
A = \begin{bmatrix} 1 & 2 \\ -5 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} -4 \\ 1 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 6 & -4 \end{bmatrix}, \quad \text{and} \quad D = [0]
\]

Verify that the system is observable and controllable. If so, design a full-state feedback law and an observer by placing the closed-loop system poles at \( s_{1,2} = -1 \pm j \) and the observer poles at \( s_{1,2} = -10 \)

Answer:

The controllability matrix is

\[
P_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 1 & 10 \end{bmatrix}
\]

And the observability matrix is

\[
P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 26 & 52 \end{bmatrix}
\]

Computing the determinants yields

\[
\det P_c = -38 \neq 0 \quad \text{and} \quad \det P_o = 416 \neq 0.
\]

Hence the system is controllable and observable. The controller gain matrix

\[
K = [3.55, 7.21]^T
\]

\[
L = q(A)P_o^{-1} \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}
\]

The observer gain matrix is
\[ L = \begin{bmatrix} 1.38 \\ -0.67 \end{bmatrix} \]