MAT 5187 Ass. n° 4 D4.2

Exercises

4.4.1. In the above section we illustrated the application of ABM methods in backward difference form by examining the case of a third-order ABM method in $P(EC)^2$ mode, and showed that the process was equivalent to that obtained by applying the methods in standard form. Do likewise for the fourth-order ABM method in PECE mode.

ABOVE SECT.

We illustrate the above procedure by considering the case of a third-order ABM method in $P(EC)^2$ mode. Recall from §3.9 that $\gamma_0^* = 1$, $\gamma_1^* = \frac{1}{2}$, $\gamma_2^* = \frac{5}{12}$ and $\gamma_3 = -\frac{1}{24}$; note that we do not need γ_0, γ_1 or γ_2. We assume that the back data $f_n^{[1]}, \nabla f_n^{[1]}$ and $\nabla^2 f_n^{[1]}$ are available. The sequence of sub-steps for the integration step from x_n to x_{n+1} is

$$\begin{array}{ll}
\text{P:} & y_{n+1}^{[0]} = y_n^{[2]} + h(f_n^{[1]} + \frac{1}{2}\nabla f_n^{[1]} + \frac{5}{12}\nabla^2 f_n^{[1]}) \\
\text{E:} & f_{n+1}^{[0]} = f(x_{n+1}, y_{n+1}^{[0]}) \\
 & \quad \nabla_0 f_{n+1}^{[1]} = f_{n+1}^{[0]} - f_n^{[1]} \\
 & \quad \nabla_0^2 f_{n+1}^{[1]} = \nabla_0 f_{n+1}^{[1]} - \nabla f_n^{[1]} \\
 & \quad \nabla_0^3 f_{n+1}^{[1]} = \nabla_0^2 f_{n+1}^{[1]} - \nabla^2 f_n^{[1]} \\
\text{C:} & y_{n+1}^{[1]} = y_{n+1}^{[0]} + \frac{5}{12} h \nabla_0^3 f_{n+1}^{[1]} \\
\text{E:} & f_{n+1}^{[1]} = f(x_{n+1}, y_{n+1}^{[1]}) \\
\text{C:} & y_{n+1}^{[2]} = y_{n+1}^{[1]} + \frac{5}{12} h (f_{n+1}^{[1]} - f_{n+1}^{[0]}) \\
 & \quad \nabla_1^3 f_{n+1}^{[1]} = \nabla_0^3 f_{n+1}^{[1]} + f_{n+1}^{[1]} - f_{n+1}^{[0]} \\
\text{Error} & T_{n+1} = -\frac{1}{24} h \nabla^3 f_{n+1}^{[1]} \\
\text{Update} & \quad \nabla f_{n+1}^{[1]} = f_{n+1}^{[1]} - f_n^{[1]} \\
 & \quad \nabla^2 f_{n+1}^{[1]} = \nabla f_{n+1}^{[1]} - \nabla f_n^{[1]}.
\end{array}$$

On expanding the differences in terms of function values, we easily find that

$$y_{n+1}^{[0]} = y_n^{[2]} + \frac{h}{12}(23 f_n^{[1]} - 16 f_{n-1}^{[1]} + 5 f_{n-2}^{[1]})$$

$$\frac{5}{12} h \nabla_0^3 f_{n+1}^{[1]} = \frac{5h}{12}(f_{n+1}^{[0]} - 3 f_n^{[1]} + 3 f_{n-1}^{[1]} - f_{n-2}^{[1]})$$

whence

$$y_{n+1}^{[1]} = y_n^{[2]} + \frac{h}{12}(5 f_{n+1}^{[0]} + 8 f_n^{[1]} - f_{n-1}^{[1]})$$

and

$$y_{n+1}^{[2]} = y_n^{[2]} + \frac{h}{12}(5 f_{n+1}^{[1]} + 8 f_n^{[1]} - f_{n-1}^{[1]}).$$

These equations will be recognized as the third-order Adams–Bashforth and Adams–Moulton methods, now in standard linear multistep form (see Table 3.2, §3.11), implemented in $P(EC)^2$ mode. Further, 14AT 5187 A21. nº 4

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5.7.4. Write the following method as a Runge–Kutta method, and find its order:

$$y_{n+2/3} = y_n + \frac{h}{3}[f(y_{n+2/3}) + f(y_n)]$$

$$y_{n+1} = y_n + \frac{h}{4}[3f(y_{n+2/3}) + f(y_n)].$$

5.12.1. Illustrate the effect of absolute stability by using the popular fourth-order explicit method (5.21) of §5.3 to compute numerical solutions of the problem $y' = Ay$, $y(0) = [1, 0, -1]^T$, where

$$A = \begin{bmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{bmatrix}$$

using two fixed steplengths, such that $\bar{h}$ is inside $\mathscr{R}_A$ for one of the values and outside it for the other.

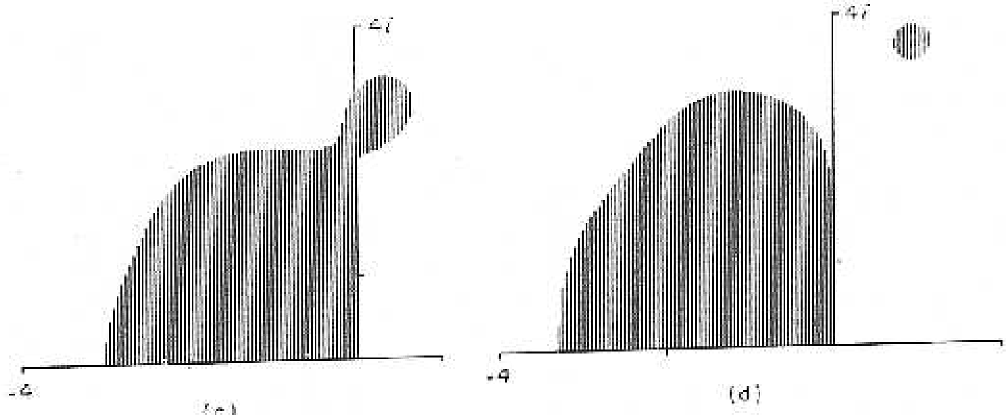

Figure 5.2 Regions of absolute stability: (a) Merson's method. (b) England's method. (c) RKF45 (d) DOPRI (5, 4).

5.12.2. Show that for all semi-implicit Runge–Kutta methods the denominator of the stability function is a product of real linear factors.

5.12.3. Convince yourself, as follows, that the 'moon' in Figure 5.2(d) is really there: using a ruler, estimate from Figure 5.2(d) the coordinates of a point inside the 'moon', and show that $|R(\hat{h})| < 1$ at that point. In a similar way, convince yourself that the 'moon' is disjoint from the main region of absolute stability.

6.4.2. Show that the method of Exercise 5.7.4 applied to the test equation $y' = \lambda y$ generates the (2, 1) Padé approximation to $\exp(h\lambda)$ and therefore cannot be A_0-stable.

6.4.3. The following method, due to Liniger and Willoughby (1970), uses the second derivatives of y, obtained by differentiating the differential system:

$$y_{n+1} - y_n = \frac{h}{2}[(1-\alpha)y^{(1)}_{n+1} + (1-\alpha)y^{(1)}_n] - \frac{h^2}{4}[(\beta+\alpha)y^{(2)}_{n+1} - (\beta-\alpha)y^{(2)}_n].$$

By an obvious extension of the definition for a linear multistep method, show that the method has order three if $\beta = \frac{1}{3}$ and order four if, in addition, $\alpha = 0$. Find the range of values for α and β for which the method is (i) A-stable and (ii) L-stable.