As per the theorem in class (3.1), we know that the greatest possible order when $k=2$ is $k+2=4$. Since we are dealing with an implicit method as well, several additional assumptions will be made.

Consider the general two-step method:

$$\alpha_2 y_{n+2} + \alpha_1 y_{n+1} + \alpha_0 y_n = h\left[\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n\right]$$

Now, we know that $\alpha_2 = 1$ & $\beta_2 \neq 0$ since the method is implicit. Following the criterion for consistency it can be said that:

$$\rho(\zeta) = \zeta^2 + \alpha_1 \zeta + \alpha_0 \qquad \sigma(\zeta) = \beta_2 \zeta^2 + \beta_1 \zeta + \beta_0$$

$$C_0 = \rho(1) = 0 \Rightarrow \boxed{\alpha_2 + \alpha_1 + \alpha_0 = 0} \; (1)$$

$$C_1 = \rho'(1) - \sigma(1) = 0 \Rightarrow \boxed{2\alpha_2 + \alpha_1 - \beta_2 - \beta_1 - \beta_0 = 0} \; (2)$$

$$C_2 = \underbrace{\frac{1}{2}\alpha_1 - \beta_1}_{j=1} + \underbrace{\frac{1}{2}(2)^2\alpha_2 - 2\beta_2}_{j=2} = 0 \Rightarrow \boxed{\frac{\alpha_1}{2} - \beta_1 + 2\alpha_2 - 2\beta_2 = 0} \; (3)$$

$$C_3 = \underbrace{\frac{1}{6}\alpha_1 - \frac{1}{2}\beta_1}_{j=1} + \underbrace{\frac{1}{6}(2)^3\alpha_2 - \frac{1}{2}(2)^2\beta_2}_{j=2} = 0 \Rightarrow \boxed{\frac{\alpha_1}{6} - \frac{\beta_1}{2} + \frac{8\alpha_2}{6} - 2\beta_2 = 0} \; (4)$$

$$C_4 = \underbrace{\frac{1}{24}\alpha_1 - \frac{1}{6}\beta_1}_{j=1} + \underbrace{\frac{1}{24}(2)^4\alpha_2 - \frac{1}{6}(2)^3\beta_2}_{j=2} = 0 \Rightarrow \boxed{\frac{\alpha_1}{24} - \frac{\beta_1}{6} + \frac{16\alpha_2}{24} - \frac{8\beta_2}{6} = 0} \; (5)$$

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Since $\alpha_2 = 1$, we can see that we have 5 eq'ns & 5 unknowns, so we can solve them:

(1) $\alpha_0 + \alpha_1 = -1$.

(2) $\alpha_1 - \beta_2 - \beta_1 - \beta_0 = -2$

(3) $\alpha_1 - 2\beta_1 - 4\beta_2 = -4$

(4) $\alpha_1 - 3\beta_1 - 12\beta_2 = -8$

(5) $\alpha_1 - 4\beta_1 - 32\beta_2 = -16$

these equ's have been simplified to remove fractions.

Equations (3), (4) & (5) can be solved first:

$$\begin{array}{cccc} \alpha_1 & \beta_1 & \beta_2 & \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & -2 & -4 & -4 \\ 1 & -3 & -12 & -8 \\ 1 & -4 & -32 & -16 \end{array}\right]} \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4/3 \\ 0 & 0 & 1 & 1/3 \end{array}\right]$$

Note: Row-reducing operations have been ommitted here.

At this point 3 parameters are known:

$\boxed{\alpha_1 = 0}$ $\boxed{\beta_1 = 4/3}$ $\boxed{\beta_2 = 1/3}$

By eq'n (1): $\boxed{\alpha_0 = -1}$

(2): $\beta_0 = 0 - 1/3 - 4/3 + 2 = 1/3 \Longrightarrow \boxed{\beta_0 = \frac{1}{3}}$

So this method is represented by:

$$y_{n+2} - y_n = \frac{h}{3}\left[f_{n+2} + 4f_{n+1} + f_n\right].$$

2/35

The method on the previous page is also known as Milne-Simpson method which is a unique method with $k=2$ & order 4.

The error constant is equal to:

$$C_5 = \underbrace{\frac{1}{120}\alpha_1 - \frac{1}{24}\beta_1}_{j=1} + \underbrace{\frac{1}{120}(2)^5\alpha_2 - \frac{1}{24}(2)^4\beta_2}_{j=2}$$

$$C_5 = 0 - \frac{1}{\cancel{24}_{6}}\cdot\left(\frac{\cancel{4}}{3}\right) + \frac{1}{120}\cdot 32 - \frac{1}{\cancel{24}_{6}}\cdot \cancel{16}^{4}\cdot\left(\frac{1}{3}\right)$$

$$C_5 = -\frac{1}{18} + \frac{4}{15} - \frac{4}{18}$$

$\boxed{C_5 = -0.011\overline{1}}$ is the error constant.

D2.4

3.2.2

A cursory examination shows this method to be consistent. Obviously the terms on the left add up to zero and

$$4 - \frac{8}{19}(2) = \frac{60}{19} = \frac{6}{19}(1+4+4+1)$$

number pp. 6, 7, ...

so $C_0 = C_1 = 0$.

Next, I put together a simple maple program (sixth page) that gives the method an order of 7, with an error coefficient of $\frac{-6}{35}$. $\times\frac{1}{19}$

Because the method is nicely symmetric around y_{n-2}, it would be easiest to work out the Taylor expansions around there. The symmetry would mean that one would need only calculate half of them, at least at the beginning.

```
> a:= [8,0,-8,19];
                              a := [8, 0, -8, 19]
> b:= [24,0,24,6];
                              b := [24, 0, 24, 6]
> for q from 2 to 7 do
  x:=sum(1/(q!)*(i^q)*(a[i]) - 1/((q-1)!)*(i^(q-1))*(b[i]),
  i=1..4):
  print(q,x);
  end do:
                                    2, 0
                                    3, 0
                                    4, 0
                                    5, 0
                                    6, 0
                                    7, -6/35
>
```

$\times\frac{1}{19}$ $C_7 = -0.00902250$

? $= -\frac{6}{665}$

your program forgot $\frac{1}{19}$

3.3.1. Newton - Gregory backward interpolation formula

$$I_q(x) = P_q(r) = \sum_{i=0}^{q} (-1)^i \binom{-r}{i} \nabla^i F_n .$$

$$I_1(x) = P_1(z) = (1 + z\nabla) f_{n+1}$$

$$y_{n+2} - y_{n+1} = \int_{x_{n+1}}^{x_{n+2}} I_1(x)\,dx = \int_{-1}^{0} (1 + z\nabla) f_{n+1} h\,dz = \int_{-1}^{0} h\,(f_{n+1} + z f_{n+1} - z f_n)\,dz =$$

$$= hz f_{n+1}\Big]_{-1}^{0} + \frac{1}{2} h z^2 f_{n+1}\Big]_{-1}^{0} - \frac{1}{2} h z^2 f_n\Big]_{-1}^{0} = h f_{n+1} + \frac{1}{2} h f_{n+1} - \frac{h f_n}{2} =$$

$= h\left(\frac{3}{2} f_{n+1} - \frac{1}{2} f_n\right)$, which is the 2-step Adams - Bashforth method.

$$I_2(x) = P_2(z) = \left(1 + z\nabla + \frac{1}{2} z(z+1)\nabla^2\right) f_{n+2}$$

$$y_{n+2} - y_{n+1} = \int_{x_{n+1}}^{x_{n+2}} h I_2(x)\,dx = \int_{-2}^{-1} h P_2(z)\,dz = \int_{-2}^{-1} h\left(1 + z\nabla + \frac{1}{2} z(z+1)\nabla^2\right) f_{n+2}\,dz =$$

$$= \int_{-2}^{-1} h\left(f_{n+2} + z\,(f_{n+2} - f_{n+1}) + \frac{1}{2} z(z+1)\,(f_{n+2} - 2f_{n+1} + f_n)\right) dz =$$

$$= \int_{-2}^{-1} h\left(f_{n+2} + z\left(f_{n+2} - f_{n+1} + \frac{1}{2}(f_{n+2} - 2f_{n+1} + f_n)\right) + \frac{1}{2} z^2 (f_{n+1} - 2f_{n+1} + f_n)\right) dz$$

$= h\left(\frac{5}{12} f_{n+2} + \frac{2}{3} f_{n+1} - \frac{1}{12} f_n\right)$, which is the 2-step Adams - Moulton method.

2nd SOLUTION

3.2.2 Find the order and error constant of

$$y_{n+4} - \frac{8}{19}(y_{n+3} - y_{n+1}) - y_n = \frac{6h}{19}(f_{n+4} + 4f_{n+3} + 4f_{n+1} + f_n)$$

Here

$\alpha_0 = -1$ $\qquad \beta_0 = 6/19$

$\alpha_1 = 8/19$ $\qquad \beta_1 = 24/19$

$\alpha_2 = 0$ $\qquad \beta_2 = 0$

$\alpha_3 = 8/19$ $\qquad \beta_3 = 24/19$

$\alpha_4 = 1$ $\qquad \beta_4 = 6/19$

$$\rho(r) = r^4 - \frac{8}{19}r^3 + \frac{8}{19}r - 1$$

$$\sigma(r) = \frac{6}{19}r^4 + \frac{24}{19}r^3 + \frac{24}{19}r + \frac{6}{19}$$

$$C_0 = \rho(1) = 1 - \frac{8}{19} + \frac{8}{19} - 1 = 0$$

$$C_1 = \rho'(1) - \sigma(1) = 4 - \frac{3\cdot 8}{19} + \frac{8}{19} - \frac{6}{19} - \frac{24}{19} - \frac{24}{19} - \frac{6}{19} = 4 - \frac{76}{19} = 0$$

$$C_2 = 0 + \left[\frac{1}{2!}\cdot\frac{8}{19} - \frac{1}{1!}\frac{24}{19}\right] + 0 + \left[\frac{1}{2!}\cdot 3^2\left(-\frac{8}{19}\right) - \frac{1}{1!}\cdot 3^1\cdot\frac{24}{19}\right]$$

$$+ \left[\frac{1}{2!}\cdot 4^2(1) - \frac{1}{1!}\cdot 4^1\cdot\frac{6}{19}\right] = \left[\frac{-20}{19}\right] + \left[\frac{-108}{19}\right] + \left[\frac{128}{19}\right] = 0$$

$$C_3 = 0 + \left[\frac{1}{3!}\cdot\frac{8}{19} - \frac{1}{2!}\cdot\frac{24}{19}\right] + 0 + \left[\frac{1}{3!}\cdot 3^3\cdot\left(\frac{-18}{19}\right) - \frac{1}{2!}\cdot 3^2\cdot\frac{24}{19}\right]$$

$$+ \left[\frac{1}{3!}\cdot 4^3\cdot(1) - \frac{1}{2!}\cdot 4^2\cdot\frac{6}{19}\right] = \left[\frac{-64}{114}\right] + \left[\frac{-864}{114}\right] + \left[\frac{928}{114}\right] = 0$$

$$C_4 = 0 + \left[\frac{1}{4!}\cdot\frac{8}{19} - \frac{1}{3!}\cdot\frac{24}{19}\right] + 0 + \left[\frac{1}{4!}\cdot 3^4\cdot\left(\frac{-8}{19}\right) - \frac{1}{3!}\cdot 3^3\cdot\frac{24}{19}\right]$$

$$+ \left[\frac{1}{4!}\cdot 4^4(1) - \frac{1}{3!}\cdot 4^3\cdot\frac{6}{19}\right] = \left[\frac{-88}{456}\right] + \left[\frac{-3240}{456}\right] + \left[\frac{3328}{456}\right] = 0$$

$$C_5 = 0 + \left[\frac{1}{5!}\cdot\frac{8}{19} - \frac{1}{4!}\cdot\frac{24}{19}\right] + 0 + \left[\frac{1}{5!}\cdot 3^5\cdot\left(\frac{-8}{19}\right) - \frac{1}{4!}\cdot 3^4\cdot\frac{24}{19}\right]$$

$$+ \left[\frac{1}{5!}\cdot 4^5(1) - \frac{1}{4!}\cdot 4^4\cdot\frac{6}{19}\right] = \left[\frac{-112}{2280}\right] + \left[\frac{-11664}{2280}\right] + \left[\frac{11776}{2280}\right] = 0$$

$$C_6 = 0 + \left[\frac{1}{6!}\cdot\frac{8}{19} - \frac{1}{5!}\cdot\frac{24}{19}\right) + 0 + \left[\frac{1}{6!}\cdot 3^6\left(\frac{-8}{19}\right) - \frac{1}{5!}\cdot 3^5\cdot\frac{24}{19}\right)$$

$$+ \left[\frac{1}{6!}\cdot 4^6(1) - \frac{1}{5!}\cdot 4^5\cdot\frac{6}{19}\right] = \left[\frac{-136}{13680}\right] + \left[\frac{-40824}{13680}\right] + \left[\frac{40960}{13680}\right] = 0$$

$$C_7 = 0 + \left[\frac{1}{7!}\cdot\frac{8}{19} - \frac{1}{6!}\cdot\frac{24}{19}\right] + 0 + \left[\frac{1}{7!}\cdot 3^7\left(\frac{-8}{19}\right) - \frac{1}{6!}\cdot 3^6\cdot\frac{24}{19}\right)$$

$$+ \left[\frac{1}{7!}\cdot 4^7(1) - \frac{1}{6!}\cdot 4^6\cdot\frac{6}{19}\right) = \left[\frac{-160}{95760}\right] + \left[\frac{-139968}{95760}\right] + \left[\frac{139264}{95760}\right] = \frac{-864}{95760}$$

$$= -\frac{6}{665} \times \frac{144}{144}$$

$$= -\frac{6}{665}$$

$$\approx -0.00902$$

Since $C_7 \neq 0$, this method has order 6.
the error constant is $C_7 \approx -0.00902$

to find the most efficient point about which to Taylor expand we look at (3.16) and at the formulae for the D_q ((23) from the online handout)

$$D_0 = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -1 + \frac{8}{19} - \frac{8}{19} + 1 = 0$$

$$D_1 = -t(-1) + (1-t)\frac{8}{19} + 0 + (3-t)\left(\frac{-8}{19}\right) + (4-t)(1) - \left(\frac{6}{19} + \frac{24}{19} + \frac{24}{19} + \frac{6}{19}\right)$$

$$= \cancel{t} + \frac{8}{19} - \cancel{\frac{8t}{19}} - \frac{24}{19} + \cancel{\frac{8t}{19}} + 4 - \cancel{t} - \frac{60}{19}$$

$$= \frac{8 - 24 + 76}{19} - \frac{60}{19} = 0$$

$$D_2 = \frac{1}{2}\left[(-t)^2(-1) + (1-t)^2\frac{8}{19} + 0 + (3-t)^2\left(\frac{-8}{19}\right) + (4-t)^2(1)\right]$$

$$-\frac{1}{1!}\left[(-t)\frac{6}{19} + (1-t)\frac{24}{19} + 0 + (3-t)\frac{24}{19} + (4-t)\frac{6}{19}\right]$$

$$= \frac{1}{2}\left[-t^2 + (1-2t+t^2)\frac{8}{19} + (9-6t+t^2)\left(\frac{-8}{19}\right) + (16-8t+t^2)\right]$$

$$-\left[-\frac{6t}{19} + \frac{24}{19} - \frac{24t}{19} + \frac{72}{19} - \frac{24t}{19} + \frac{24}{19} - \frac{6t}{19}\right]$$

$$= \cancel{\frac{-t^2}{2}} + \frac{4}{19} - \frac{8t}{19} + \cancel{\frac{4t^2}{19}} - \frac{36}{19} + \frac{24t}{19} - \cancel{\frac{4t^2}{19}} + 8 - 4t + \cancel{\frac{t^2}{2}} - \frac{60t}{19} + \frac{120}{19}$$

$$= -\frac{120t}{19} + \frac{240}{19} = 0 \quad \Rightarrow \quad \boxed{t = 2}$$

Now checking $D_3 \to D_6$ using $t = 2$

$$D_3 = \frac{1}{3!}\left[(-2)^3(-1) + (-1)^3\frac{8}{19} + (1)^3\left(\frac{-8}{19}\right) + 2^3(1)\right] - \frac{1}{2!}\left[(-2)^2\frac{6}{19} + (-1)^2\frac{24}{19} + 1^2\frac{24}{19} + 2^2\frac{6}{19}\right]$$

$$= \left(\frac{48}{19}\right) - \left[\frac{48}{19}\right] = 0$$

$$D_4 = \frac{1}{4!}\left[(-2)^4(-1) + (-1)^4\frac{8}{19} + (1)^4\left(\frac{-8}{19}\right) + 2^4(1)\right] - \frac{1}{3!}\left[(-2)^3\cancel{\frac{6}{19}} + (-1)^3\cancel{\frac{24}{19}} + (1)^3\cancel{\frac{24}{19}} + (2)^3\cancel{\frac{6}{19}}\right]$$

$$= \frac{-304 + 8 - 8 + 304}{456} = 0$$

$$D_5 = \frac{1}{5!}\left[(-2)^5(-1) + (-1)^5\frac{8}{19} + (1)^5\left(\frac{-8}{19}\right) + 2^5(1)\right] - \frac{1}{4!}\left[(-2)^4\frac{6}{19} + (-1)^4\frac{24}{19} + (1)^4\frac{24}{19} + (2)^4\frac{6}{19}\right]$$

$$= \left[\frac{10}{19}\right] - \left[\frac{10}{19}\right] = 0$$

D 2.9

$$D_6 = \frac{1}{6!}\left[(-2)^6(-1)+(-1)^6\frac{8}{19}+(1)^6\left(\frac{-8}{19}\right)+2^6(1)\right]-\frac{1}{5!}\left[\cancel{(-2)^5\frac{6}{19}}+\cancel{(-1)^5\frac{24}{19}}+\cancel{(1)^5\frac{24}{19}}+\cancel{2^5\frac{6}{19}}\right]$$

$$= \frac{1}{720}\left[\cancel{-64}+\cancel{\frac{8}{19}}-\cancel{\frac{8}{19}}+\cancel{64}\right] = 0$$

$$D_7 = \frac{1}{7!}\left[(-2)^7(-1)+(-1)^7\frac{8}{19}+(1)^7\left(\frac{-8}{19}\right)+2^7(1)\right]-\frac{1}{6!}\left[(-2)^6\frac{6}{19}+(-1)^6\frac{24}{19}+(1)^6\frac{24}{19}+2^6\cdot\frac{6}{19}\right]$$

$$= \frac{1}{5040}\left[128-\frac{8}{19}-\frac{8}{19}+128\right]-\frac{1}{720}\left[\frac{64\cdot 6}{19}+\frac{24}{19}+\frac{24}{19}+\frac{64\cdot 6}{19}\right]$$

$$= \frac{4848}{95760}-\frac{816}{13680} \approx -0.00902$$ same as before

So the best value around which to expand the taylor series is according to (3.16)

$$x+th = \boxed{x+2h}$$

3.5.1 verify (3.25) for $y' = \lambda y$, $y(0) = 1$
using i) Euler's method, ii) Trapezoidal method.

(3.25) $[I - h\beta_k J(x_{n+k}, \eta_{n+k})][y(x_{n+k}) - \tilde{y}_{n+k}] = T_{n+k}$

i) Euler's method: $y_{n+1} - y_n = h f_n$
here $k=1$ $\alpha_0 = -1$ $\beta_0 = 1$
$\alpha_1 = 1$ $\beta_1 = 0$

First find T_{n+1} using (3.23) $T_{n+k} = \mathcal{L}[y(x_n); h]$
where $y(x_n)$ is the exact solution.
$y(x_n) = e^{\lambda x_n}$, $y'(x_n) = \lambda e^{\lambda x_n}$

So $T_{n+1} = \mathcal{L}[y(x_n); h] = \sum_{j=0}^{1} [\alpha_j y(x+jh) - h\beta_j y'(x+jh)]$

$= (-1) y(x_n) - h y'(x_n) + y(x_n + h)$

$= -e^{\lambda x_n} - h\lambda e^{\lambda x_n} + e^{\lambda(x_n+h)}$

$= e^{\lambda x_n}(-1 - h\lambda + e^{\lambda h})$ where $x_n = nh$

$= e^{\lambda nh}(-1 - h\lambda + e^{\lambda h})$

Now in (3.25) $\beta_k = \beta_1 = 0$ So

$T_{n+1} = [y(x_{n+1}) - \tilde{y}_{n+1}] = y_{n+1} - \tilde{y}_{n+1}$

on p. 56 $\tilde{y}$ is given by:

$\tilde{y}_{n+1} - y_n = h f_n$

$\Leftrightarrow$ $y_n - \tilde{y}_{n+1} = -h f_n$

$\Leftrightarrow$ $y_{n+1} - \tilde{y}_{n+1} = -h f_n - y_n + y_{n+1}$ ← this should be T_{n+1}

$-h f_n - y_n + y_{n+1} = -h y'(x_n) - y(x_n) + y(x_{n+1})$

$= -h\lambda e^{\lambda x_n} - e^{\lambda x_n} + e^{\lambda x_{n+1}}$ now use $x_n = nh$

$= -h\lambda e^{\lambda nh} - e^{\lambda nh} + e^{\lambda(n+1)h}$

$= e^{\lambda nh}(-h\lambda - 1 + e^{\lambda h}) = T_{n+1}$ as in (3.23)

ii) the Trapezoidal Rule: $y_{n+1} - y_n = \frac{h}{2}(f_{n+1} + f_n)$

here $k=1$ $\quad \alpha_0 = -1 \quad \beta_0 = \frac{1}{2}$
$\quad \alpha_1 = 1 \quad \beta_1 = \frac{1}{2}$

from (3.23)

$$T_{n+1} = \sum_{j=0}^{1} [\alpha_j\, y(x_n + jh) - \beta_j\, y'(x_n + jh)]$$

$$= -1\, y(x_n) - \frac{h}{2} y'(x_n) + y(x_n + h) - \frac{h}{2} y'(x_n + h)$$

$$= -e^{\lambda x_n} - \frac{h\lambda}{2} e^{\lambda x_n} + e^{\lambda(x_n + h)} - \frac{h\lambda}{2} e^{\lambda(x_n + h)}$$

$$= e^{\lambda x_n}\left[-1 - \frac{h\lambda}{2} + e^{\lambda h} - \frac{h\lambda}{2} e^{\lambda h}\right] \quad x_n = nh \text{ so}$$

$$= e^{\lambda n h}\left[(e^{\lambda h} - 1) - \frac{h\lambda}{2}(1 + e^{\lambda h})\right]$$

Now to check (3.25) Note that $\bar{J}(x_{n+1}, \eta_{n+1}) = 0$ since it is the jacobian of f with respect to y and f depends only on x so

(3.25) $[I - h\beta_1 \bar{J}(x_{n+1}, \eta_{n+1})][y(x_{n+1}) - \tilde{y}_{n+1}]$

$$= y(x_{n+1}) - \tilde{y}_{n+1} = y_{n+1} - \tilde{y}_{n+1} = T_{n+1}$$

Again from p. 56

$$\tilde{y}_{n+1} - y_n = \frac{h}{2} f(x_{n+1}, \tilde{y}_{n+1}) + \frac{h}{2} f(x_n, y_n)$$

$$\Rightarrow \quad \tilde{y}_{n+1} = y_n + \frac{h}{2}(f_{n+1} + f_n)$$

$$\Rightarrow \quad y_{n+1} - \tilde{y}_{n+1} = y_{n+1} - y_n - \frac{h}{2}(f_{n+1} + f_n)$$

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D2.12

$$y_{n+1} - \tilde{y}_{n+1} = y_{n+1} - y_n - \frac{h}{2}(f_{n+1} + f_n)$$

$$= y_{n+1} - y_n - \frac{h}{2} y'_{n+1} - \frac{h}{2} y'_n$$

$$= e^{\lambda x_{n+1}} - e^{\lambda x_n} - \frac{h\lambda}{2} e^{\lambda x_{n+1}} - \frac{h\lambda}{2} e^{\lambda x_n}$$ sub in $x_n = nh$

$$= e^{\lambda(n+1)h} - e^{\lambda nh} - \frac{h\lambda}{2} e^{\lambda(n+1)h} - \frac{h\lambda}{2} e^{\lambda nh}$$

$$= e^{\lambda nh}\left(e^{\lambda h} - 1 - \frac{h\lambda}{2} e^{\lambda h} - \frac{h\lambda}{2}\right)$$

$$= e^{\lambda nh}\left[\left(e^{\lambda h} - 1\right) - \frac{h\lambda}{2}\left(e^{\lambda h} + 1\right)\right]$$ $= T_{n+1}$ as before

✓

3.5.2.

$$y_{n+2} - (1+\alpha) y_{n+1} + \alpha y_n = h \left[(1+\beta) f_{n+2} - (\alpha + \beta + \alpha\beta) f_{n+1} + \alpha \beta f_n \right.$$

with $y' = y$, $y(0) = 1$ $\Rightarrow$ $y = e^x$

has exact solution (from 2.5.5*)

$$y_n = \left[A\alpha^n + B\left(\frac{1-\beta h}{1-(1+\beta)h}\right)^n \right] \Big/ G$$

where

$A = (-1+\beta h)\eta_0 + [1-(1+\beta)h]\eta_1$

$B = [1-(1+\beta)h](\alpha\eta_0 - \eta_1)$

$G = \alpha - 1 - (\alpha - \beta + \alpha\beta)h$

From the results on p. 50 we need only to consider 2 cases:

i) if $\alpha \neq 1$, $\beta \neq -\frac{1}{2}$, $p = 1$ error constant $C_2 = (\alpha-1)(\beta+\frac{1}{2})$

ii) if $\alpha \neq 1$, $\beta = -\frac{1}{2}$, $p = 2$ error constant $C_3 = \frac{1}{12}(\alpha - 1)$

the other 2 case case have $\alpha = 1$, and the method diverges.

i) the LTE is given by:

$$\text{LTE} = T_n = C_2 h^2 y^{(2)}(x_n) + O(h^3)$$
$$= (\alpha-1)(\beta+\tfrac{1}{2}) e^{x_n} h^2 + O(h^3)$$

To find the GTE we need to simplify the expression for y_n. We will use starting values $\eta_0 = 1$, and $\eta_1 = \dfrac{(1-\beta h)}{[1-(1+\beta)h]}$ (see 2.5.5* part ii)

then $A = -1 + \beta h + 1 - \beta h = 0$

$$B = [1-(1+\beta)h]\left(\alpha - \frac{(1-\beta h)}{[1-(1+\beta)h]}\right)$$
$$= [1-(1+\beta)h]\alpha - 1 + \beta h$$

$$B = [1-(1+\beta)h][\alpha - \frac{(1-\beta h)}{1-(1+\beta)h}]$$
$$= \alpha - \alpha(1+\beta)h - (1-\beta h)$$
$$= (\beta - \alpha - \alpha\beta)h + \alpha - 1 = \alpha - 1 - (\alpha - \beta + \alpha\beta)h = d$$

then

$$y_n = \left(\frac{1-\beta h}{1-(1+\beta)h}\right)^n = \begin{cases} e^{x_n}[1+(\tfrac{1}{2}+\beta)x_n h + O(h^2)] & \text{if } \beta \neq \\ e^{x_n}[1+\tfrac{1}{12}x_n h^2 + O(h^3)] & \text{if } \beta = -\tfrac{1}{2} \end{cases}$$

So in i) $\alpha \neq 1$ $\beta \neq 1$ $p=1$

$$LTE = (\alpha-1)(\beta+\tfrac{1}{2})e^{x_n}h^2 + O(h^3)$$
$$GTE = y(x_n) - y_n$$
$$= e^{x_n} - e^{x_n}[1+(\tfrac{1}{2}+\beta)x_n h + O(h^2)]$$
$$= (\tfrac{1}{2}+\beta)x_n h + O(h^2)$$

$\Rightarrow$ We lose a power of h in the GTE which is the same result as in the example on p. 57-58

and in ii) $\alpha \neq 1$ $\beta = -\tfrac{1}{2}$ $p=2$

$$LTE = C_3 h^3 y^{(3)}(x_n) + O(h^4)$$
$$= \tfrac{1}{12}(\alpha-1)e^{x_n}h^3 + O(h^4)$$
$$GTE = y(x_n) - y_n$$
$$= e^{x_n} - e^{x_n}[1+\tfrac{1}{12}x_n h^2 + O(h^3)]$$
$$= \tfrac{1}{12}x_n h^2 + O(h^3)$$

$\Rightarrow$ Again we have lost a power of h a

3.8.1 $y_{n+2} - y_n = \frac{1}{2} h (f_{n+1} + 3 f_n)$

$\rho(r) = r^2 - 1$

$\sigma(r) = \frac{1}{2}(r+3)$

$$\hat{h}(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} = \frac{e^{2i\theta} - 1}{\frac{1}{2}(e^{i\theta}+3)}$$

We have to show that this (equation of) (function describes the) circle with center at $(-\frac{2}{3}, 0)$ and radius $\frac{2}{3}$.
We have to show that

$$\left| \hat{h}(\theta) + \frac{2}{3} \right| = \frac{2}{3}$$

$$\left| \hat{h}(\theta) + \frac{2}{3} \right| = \left| \frac{e^{2i\theta} - 1}{\frac{1}{2}(e^{i\theta}+3)} + \frac{2}{3} \right| = \left| \frac{6e^{2i\theta} - \cancel{6} + 2e^{i\theta} + \cancel{6}}{3(e^{i\theta}+3)} \right| =$$

$$= \left| \frac{2e^{i\theta}(3e^{i\theta}+1)}{3(e^{i\theta}+3)} \right| = \underbrace{\left| \frac{2}{3} \right|}_{2/3} \underbrace{\left| e^{i\theta} \right|}_{1} \left| \frac{3e^{i\theta}+1}{e^{i\theta}+3} \right| =$$

$$= \frac{2}{3} \left| \frac{e^{i\theta}(3 + e^{-i\theta})}{e^{i\theta}+3} \right| = \frac{2}{3} \left| \frac{e^{-i\theta}+3}{e^{i\theta}+3} \right| = \frac{2}{3} \left| \frac{3 + \cos(-\theta) + i\sin(-\theta)}{3 + \cos(\theta) + i\sin(\theta)} \right| =$$

$$= \frac{2}{3} \left| \frac{3 + \cos\theta - i\sin\theta}{3 + \cos\theta + i\sin\theta} \right| = \frac{2}{3}$$

conjugate so the module is equal
$|\alpha + i\beta| = |\alpha - i\beta|$

Routh-Hurwitz

$$\pi(r, \hat{h}) = r^2 - 1 - \frac{\hat{h}}{2}(r+3) = r^2 - 1 - Hr - 3H \qquad H = \frac{\hat{h}}{2}$$

$r = \frac{1+z}{1-z}$

$$(1-z)^2 \cdot \Pi\left({}^{1+z}/_{1-z} ; \hat{h}\right) = (1-z)^2\left(\frac{1+z}{1-z}\right)^2 - H(1-z)^2\left(\frac{1+z}{1-z}\right)(3H+1)(1-z)^2 =$$

$$= (1+z)^2 - H(1-z)(1+z) - (3H+1)(1-z^2) =$$

$$= 1+2z+z^2 - H - Hz^2 - (3H+1)(1-2z+z^2) =$$

$$= \cancel{1} + 2z + \cancel{z^2} - H - Hz^2 - 3H + 6Hz - 3Hz^2 - \cancel{1} + 2z - \cancel{z^2} =$$

$$= -4Hz^2 + (4+6H)z - 4H$$

$a_2 = -4H > 0 \quad \Rightarrow \quad H < 0$

$a_1 = 4+6H > 0 \quad \Rightarrow \quad 6H > -4 \qquad H > -2/3$

$a_0 = -4H > 0 \quad \Rightarrow \quad H < 0$

$$-2/3 < H < 0$$

$$-2/3 < \frac{\hat{h}}{2} < 0$$

$$-4/3 < \hat{h} < 0$$

3.8.2 3step Adams Multon applied to scalar problem

$y''+20y'+200y=0 \quad y(0)=1 \quad y'(0)=-10$

$u_1=y$

$u_2=y'$

$u_2'=y''=-20y'-200y=-20u_2-200u_1$

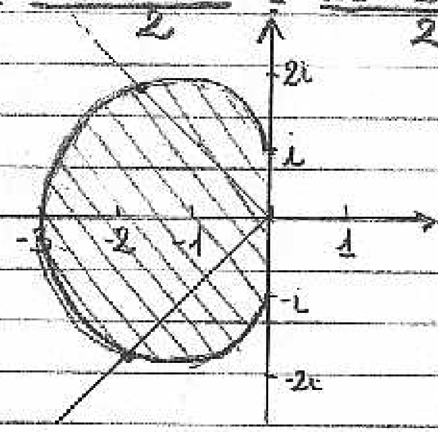
$$\overset{y'}{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}'} \overset{=}{=} \overset{A}{\begin{bmatrix} 0 & 1 \\ -200 & -20 \end{bmatrix}} \overset{y}{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}$$

$$p(\lambda)=\det(A-\lambda I)=\det\begin{bmatrix} -\lambda & 1 \\ -200 & -20-\lambda \end{bmatrix}=$$

$$=-\lambda(-20-\lambda)+200=\lambda^2+20\lambda+200$$

$\Delta=400-800=-400$

$$\lambda_{1,2}=\frac{-20\pm\sqrt{-400}}{2}=\frac{-20\pm 2i\sqrt{100}}{2}=-10\pm 10i$$



$|-10\pm10i|=\sqrt{10^2+10^2}=\sqrt{200}\approx$
$\approx 14{,}14$

$|\lambda|h <\sim 2{,}5$

$14{,}14\,h<2{,}5 \Rightarrow h<0{,}18$

$$[A-\lambda_1 I]v=\begin{bmatrix} 10+10i & 1 \\ -200 & -20+10+10i \end{bmatrix}v=\begin{bmatrix} 10+10i & 1 \\ -200 & -10+10i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=0$$

$$\begin{cases} (10+10i)v_1+v_2=0 \\ (-200)v_1+(-10+10i)v_2=0 \end{cases}$$

D 2. 18

$$v_2 = (-10-10i)v_1$$

$$-200v_1 + (-10+10i)(-10-10i) = 0$$
$$-200v_1 + 100 + 100 = 0$$
$$200v_1 = 200$$
$$v_1 = 1 \implies v_2 = -10-10i$$

$$v = \begin{bmatrix} 1 \\ -10-10i \end{bmatrix} = \begin{bmatrix} 1 \\ -10 \end{bmatrix} + i \begin{bmatrix} 0 \\ -10 \end{bmatrix}$$

$$e^{\lambda x} v = e^{(-10-10i)x}\left(\begin{bmatrix} 1 \\ -10 \end{bmatrix} + i \begin{bmatrix} 0 \\ -10 \end{bmatrix}\right) =$$

$$= e^{-10x} e^{-10ix}\left(\begin{bmatrix} 1 \\ -10 \end{bmatrix} + i \begin{bmatrix} 0 \\ -10 \end{bmatrix}\right) = e^{-10x}(\cos -10x + i \sin -10x)\left(\begin{bmatrix} 1 \\ -10 \end{bmatrix} + i \begin{bmatrix} 0 \\ -10 \end{bmatrix}\right) =$$

$$= e^{-10x}(\cos 10x - i \sin 10x)\left(\begin{bmatrix} 1 \\ -10 \end{bmatrix} + i \begin{bmatrix} 0 \\ -10 \end{bmatrix}\right) =$$

$$= e^{-10x}\left(\begin{bmatrix} \cos 10x \\ -10\cos 10x - 10\sin 10x \end{bmatrix} + i \begin{bmatrix} -\sin 10x \\ 10\sin 10x - 10\cos 10x \end{bmatrix}\right)$$

$$\begin{bmatrix} y \\ y' \end{bmatrix} = u = c_1 e^{-10x} \begin{bmatrix} \cos 10x \\ -10\cos 10x - 10\sin 10x \end{bmatrix} + c_2 e^{-10x} \begin{bmatrix} -\sin 10x \\ 10\sin 10x - 10\cos 10x \end{bmatrix}$$

$$1 = c_1 e^0 \cos 0 + c_2 e^0 (-\sin 0) = c_1$$

$$-10 = c_1 e^0(-10\cos 0 - 10\sin 0) + c_2 e^0 (10\sin 0 - 10\cos 0) =$$

$$= -10c_1 - 10c_2 = -10 - 10c_2$$

$$1 = 1 + c_2 \implies c_2 = 0$$

D 2.19

$$\begin{bmatrix} y \\ y' \end{bmatrix} = e^{-10x} \begin{bmatrix} \cos 10x \\ -10\cos 10x - 10\sin 10x \end{bmatrix}$$

3 step AM method

$$y_{n+1} - y_n = \frac{h}{24}\left(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right)$$

for our system $\begin{bmatrix} y \\ y' \end{bmatrix}' = \underset{\substack{\uparrow \\ f}}{\begin{bmatrix} y' \\ -200y - 20y' \end{bmatrix}}$

$$\begin{bmatrix} y_{n+1}^{[v+1]} \\ y'^{[v+1]}_{n+1} \end{bmatrix} = \begin{bmatrix} y_n \\ y'_n \end{bmatrix} + \frac{h}{24}\begin{bmatrix} 0 & 1 \\ -200 & -20 \end{bmatrix}\left(9\begin{bmatrix} y_{n+1}^{[v]} \\ y'^{[v]}_{n+1} \end{bmatrix} + 19\begin{bmatrix} y_n \\ y'_n \end{bmatrix} - 5\begin{bmatrix} y_{n-1} \\ y'_{n-1} \end{bmatrix} + \begin{bmatrix} y_n \\ y'_n \end{bmatrix}\right.$$

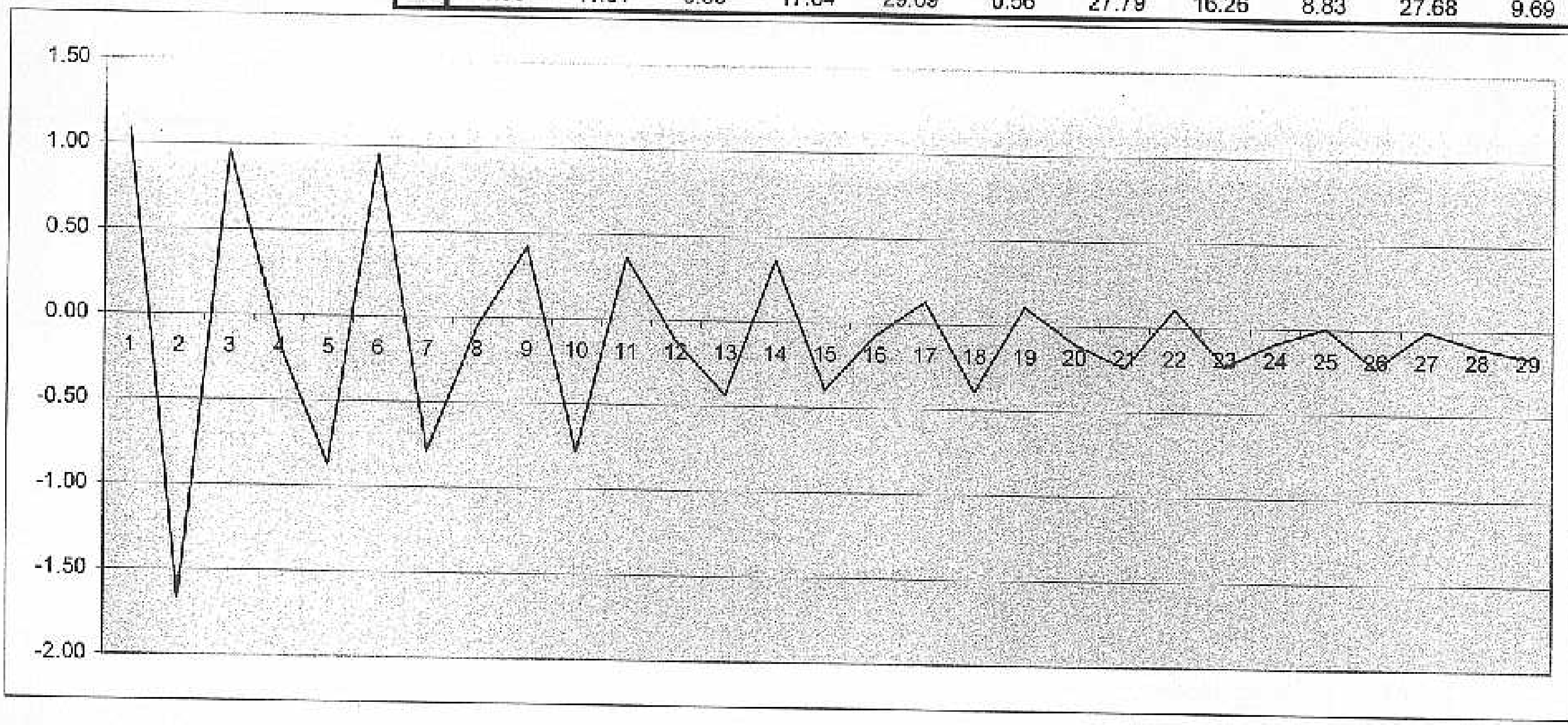
we take $y_0, y'_0, y_1, y'_1, y_2, y'_2, y_3^{[0]}, y'^{[0]}_3$ from exact solution and calculate $y_3^{[1]}$ and $y'^{[1]}_3$

D 2.20

h = 0.17 inside stability region, solution converges to exact one (err → 0)

	h=	0.17		
-2	x=	-0.34	x=	-0.34
	y=	######	y'=	213.12
-1	x=	-0.17	x=	-0.17
	y=	-0.71	y'=	61.34
0	x=	0.00	x=	0.00
	y=	1.00	y'=	-10.00
1	x=	0.17	x=	0.17
	y=	-0.02	y'=	-1.58

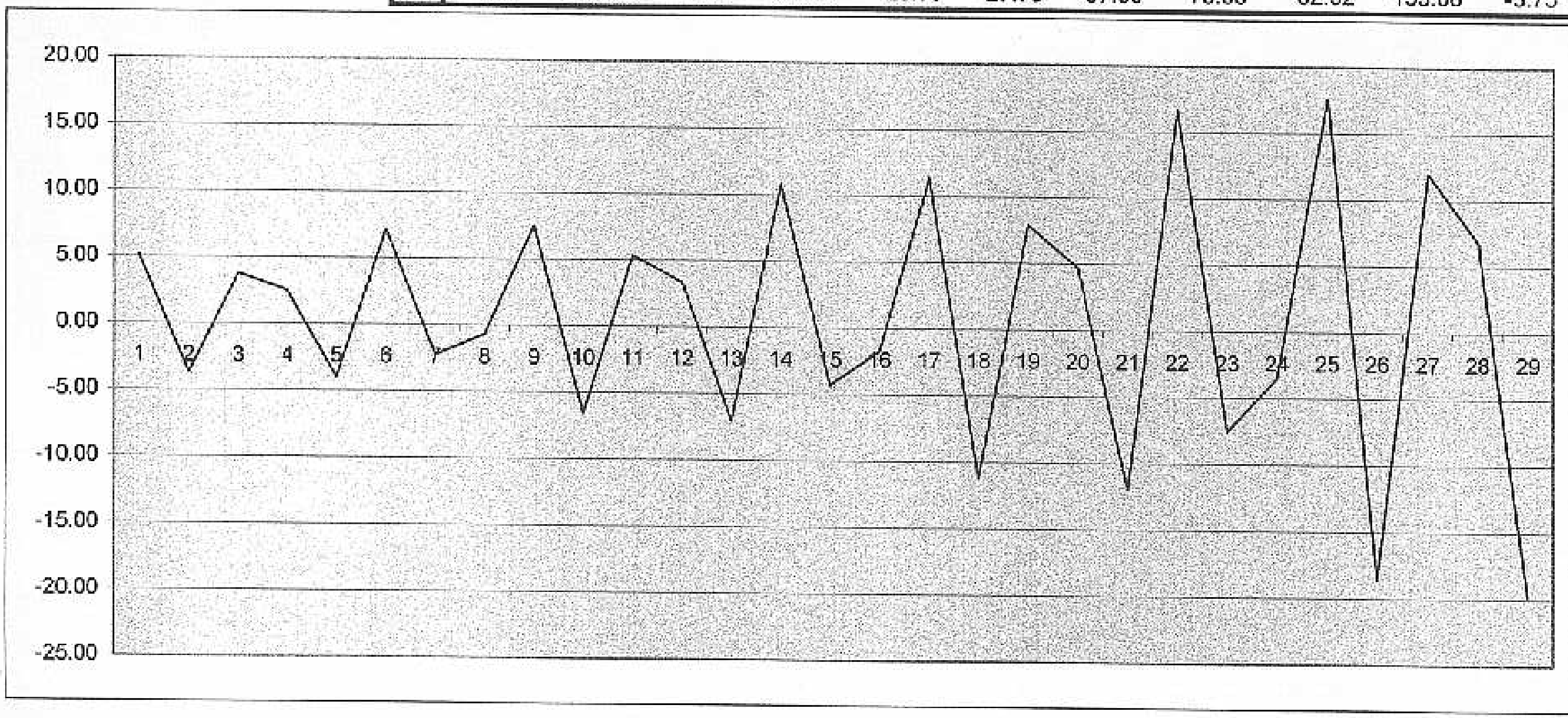
y	v=0	v=1	v=2	...							
n-2	-28.97	-28.97	-28.97	-28.97	-28.97	-28.97	-28.97	-28.97	-28.97	-28.97	-28.97
n-1	-0.71	-0.71	-0.71	-0.71	-0.71	-0.71	-0.71	-0.71	-0.71	-0.71	-0.71
n	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
n+1	-0.02	-1.11	1.64	-0.98	0.13	0.85	-0.97	0.76	0.03	-0.45	0.76
err	0.00	1.09	-1.67	0.96	-0.15	-0.87	0.95	-0.79	-0.05	0.42	-0.78
y'											
n-2	213.12	213.12	213.12	213.12	213.12	213.12	213.12	213.12	213.12	213.12	213.12
n-1	61.34	61.34	61.34	61.34	61.34	61.34	61.34	61.34	61.34	61.34	61.34
n	-10.00	-10.00	-10.00	-10.00	-10.00	-10.00	-10.00	-10.00	-10.00	-10.00	-10.00
n+1	-1.58	41.61	0.39	17.84	29.09	0.56	27.79	16.26	8.83	27.68	9.69



h-0.20 outside stability region, solution diverges

	h=	0.2		
-2	x=	-0.40	x=	-0.40
	y=	#####	y'=	-56.32
-1	x=	-0.20	x=	-0.20
	y=	-3.07	y'=	97.94
0	x=	0.00	x=	0.00
	y=	1.00	y'=	-10.00
1	x=	0.20	x=	0.20
	y=	-0.06	y'=	-0.67

y	v=0	v=1	v=2	...							
n-2	-35.69	-35.69	-35.69	-35.69	-35.69	-35.69	-35.69	-35.69	-35.69	-35.69	-35.69
n-1	-3.07	-3.07	-3.07	-3.07	-3.07	-3.07	-3.07	-3.07	-3.07	-3.07	-3.07
n	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
n+1	-0.06	-5.18	3.62	-3.82	-2.57	3.92	-7.22	2.19	0.61	-7.60	6.50
err	0.00	5.13	-3.68	3.76	2.51	-3.98	7.16	-2.25	-0.67	7.55	-6.55
y'											
n-2	-56.32	-56.32	-56.32	-56.32	-56.32	-56.32	-56.32	-56.32	-56.32	-56.32	-56.32
n-1	97.94	97.94	97.94	97.94	97.94	97.94	97.94	97.94	97.94	97.94	97.94
n	-10.00	-10.00	-10.00	-10.00	-10.00	-10.00	-10.00	-10.00	-10.00	-10.00	-10.00
n+1	-0.67	116.70	17.55	34.24	120.76	-27.79	97.69	76.58	-32.92	155.08	-3.73



D2.21