Take home exam in MAT 5187
Due Wednesday on the 25th of July 2007

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Question 1: Show that convergence implies consistency of numerical methods for totally stable ode's.

Question 2: Lambert's 1991, p. 51, exercise 3.2.5*. Suppose that the order of y_{n+r} is sufficiently high.

Question 3: Prove that Matlab's `ode23` is made of a method of **exact** order 3 to advance the step with y_{n+1} and a method of **exact** order 2 to monitor the local truncation error with $\widehat{y}_{n+1}$, by verifying that these methods satisfy the appropriate order conditions. The Butcher tableau is:

	$\mathbf{c}$	A			
k_1	0	0			
k_2	$\frac{1}{2}$	$\frac{1}{2}$			
k_3	$\frac{3}{4}$	0	$\frac{3}{4}$		
k_4	1	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	
y_{n+1}	$\mathbf{b}^T$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	0
$\widehat{y}_{n+1}$	$\widehat{\mathbf{b}}^T$	$\frac{7}{24}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{8}$
EE	$\mathbf{b}^T - \widehat{\mathbf{b}}^T$	$-\frac{5}{72}$	$\frac{1}{12}$	$\frac{1}{9}$	$-\frac{1}{8}$

Description of the method. The code `ode23` consists of a four-stage pair of embedded explicit Runge–Kutta methods of orders 2 and 3 with error control. It advances the solution from y_n to y_{n+1} with the third-order method (so called local extrapolation) and controls the local error by taking the difference between the third-order and the second-order numerical solutions.

The four stages are:

$$\begin{aligned}
k_1 &= h\,f(x_n, y_n),\\
k_2 &= h\,f(x_n + (1/2)h, y_n + (1/2)k_1),\\
k_3 &= h\,f(x_n + (3/4)h, y_n + (3/4)k_2),\\
k_4 &= h\,f(x_n + h, y_n + (2/9)k_1 + (1/3)k_2 + (4/9)k_3),
\end{aligned}$$

The first three stages produce the order-3 solution, at the next time step:

$$y_{n+1} = y_n + \frac{2}{9}\,k_1 + \frac{1}{3}\,k_2 + \frac{4}{9}\,k_3,$$

and all four stages give the local error estimate:

$$E = -\frac{5}{72}\,k_1 + \frac{1}{12}\,k_2 + \frac{1}{9}\,k_3 - \frac{1}{8}\,k_4.$$

However, this is really a three-stage method since the first step at x_{n+1} is the same as the last step at x_n, that is $k_1^{[n+1]} = k_4^{[n]}$. Such methods are called FSAL methods.

The natural interpolant used in `ode23` is the two-point Hermite polynomial of degree 3 which interpolates y_n and $f(x_n, y_n)$ at $x = x_n$, and y_{n+1} and $f(x_{n+1}, x_{n+1})$ at $t = x_{n+1}$.

Question 4: Prove that k-step BDF methods of order k are unstable for $k \geq 7$.

A short proof is found in E. Hairer and G. Wanner, *On the instability of the BDF formulae*, SIAM J. Numer. Anal., **20**(6) (1983), 1206–1209. Can be downloaded from the course description in my home page: `http://www.site.uottawa.ca/~remi`