T2.1

MID-TERM 2 MAT 3380

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Q 1) $L = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 13 \\ 1 \end{bmatrix}$

$Ly = b$

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 13 \\ 1 \end{bmatrix}$$

$\Rightarrow y_1 = 1$

$3y_1 + y_3 = 13 \Rightarrow y_3 = 13 - 3(1)$
$= 10$

$2y_1 + y_2 = 1 \Rightarrow y_2 = 1 - 2(1)$
$y_2 = -1$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 10 \end{bmatrix}$$

$Ux = y$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 10 \end{bmatrix}$$

$5x_3 = 10 \Rightarrow x_3 = 2$

$3x_2 + x_3 = -1$

$\Rightarrow x_2 = \dfrac{-1 - x_3}{3}$

$x_2 = \dfrac{-1 - (2)}{3} = -1$

$2x_1 + x_2 = 1$

$x_1 = \dfrac{1 - x_2}{2} = \dfrac{1 - -1}{2} = 1$

Therefore $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

**Q2)** Find the Cholesky decomposition and compute the <u>determinant</u> of the matrix

$L = GG^T$ where

$$G = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$$

$$L = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 10 & 1 \\ -2 & 1 & 21 \end{bmatrix}$$

$$\begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} g_{11} & g_{21} & g_{31} \\ 0 & g_{22} & g_{32} \\ 0 & 0 & g_{33} \end{bmatrix} = \begin{bmatrix} 4 & -2 & -2 \\ -2 & 10 & 1 \\ -2 & 1 & 21 \end{bmatrix}$$

$g_{11}^2 = 4 > 0 \Rightarrow g_{11} = 2 > 0$

$g_{11} g_{21} = -2 \Rightarrow g_{21} = \frac{-2}{2} = -1$

$g_{11} g_{31} = -2 \Rightarrow g_{31} = \frac{-2}{2} = -1$

$g_{21}^2 + g_{22}^2 = 10 \Rightarrow g_{22}^2 = 10 - (-1)^2 = 9 > 0 \qquad g_{22} = 3 > 0$

$g_{31} g_{21} + g_{32} g_{22} = 1 \Rightarrow g_{32} g_{22} = 1 - (-1)(-1)$

$\Rightarrow g_{32} = \frac{0}{3} = 0$

$g_{31}^2 + g_{32}^2 + g_{33}^2 = 21 \qquad g_{33}^2 = 21 - g_{31}^2 - g_{32}^2$

$= 21 - (-1)^2 - 0^2$

$= 20 > 0$

$g_{33} = 4.472 > 0$

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$$G = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ -1 & 0 & 4.472 \end{bmatrix}$$

$\longrightarrow$ ∂

the det of L

$$\det L = \det(GG^T) = \det G \det G^T$$
$$= (\det G)(\det G)$$
$$= (\det G)^2$$
$$= (2 \times 3 \times 4.472)^2 \quad \text{(because triangular)}$$
$$= 719.96$$
$$720$$

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Q3) Gauss-Seidel

$$6x_1 + x_2 - x_3 = 3 \qquad \text{with } x_1^{(0)} = 1$$
$$-x_1 + x_2 + 7x_3 = -17 \qquad x_2^{(0)} = 1$$
$$x_1 + 5x_2 + x_3 = 0 \qquad x_3^{(0)} = 1$$

rearrange

$$6x_1 + x_2 - x_3 = 3$$
$$x_1 + 5x_2 + x_3 = 0$$
$$-x_1 + x_2 + 7x_3 = -17$$

the scheme is

$$x_1^{(m+1)} = \frac{1}{6}\left[3 - x_2^{(m)} + x_3^{(m)}\right]$$
$$x_2^{(m+1)} = \frac{1}{5}\left[0 - x_1^{(m+1)} - x_3^{(m)}\right]$$
$$x_3^{(m+1)} = \frac{1}{7}\left[-17 + x_1^{(m+1)} - x_2^{(m+1)}\right]$$

$m=0$ one iteration

$$x_1^{(1)} = \frac{1}{6}\left[3 - x_2^{(0)} + x_3^{(0)}\right] = \frac{1}{6}\left[3 - 1 + 1\right] = 0.5$$

$$x_2^{(1)} = \frac{1}{5}\left[0 - x_1^{(1)} - x_3^{(0)}\right] = \frac{1}{5}\left[0 - 0.5 - 1\right] = -0.3$$

$$x_3^{(1)} = \frac{1}{7}\left[-17 + x_1^{(1)} - x_2^{(1)}\right] = \frac{1}{7}\left[-17 + 0.5 - (-0.3)\right]$$

$$= -2.314$$

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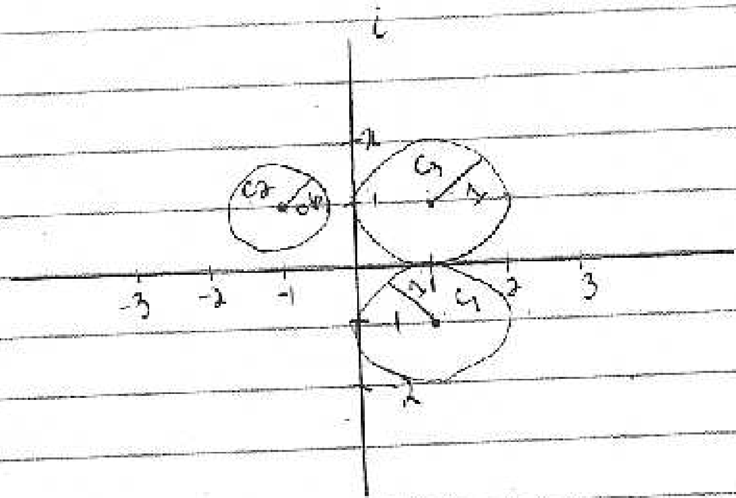
Q4) $\begin{bmatrix} 1-i & 0.3+0.4i & 0.5i \\ 0.3i & -1+i & 0.3 \\ 0 & 0.6+0.8i & 1+i \end{bmatrix}$

$z = x+yi$
$|z| = \sqrt{x^2+y^2}$

$c_1 = 1-i \qquad r_1 = |0.3+0.4i| + |0.5i| = \sqrt{(0.3)^2+(0.4)^2} + 0.5 = 1$

$c_2 = -1+i \qquad r_2 = |0.3i| + 0.3 = 0.6$

$c_3 = 1+i \qquad r_3 = |0| + |0.6+0.8i| = \sqrt{0.6^2+0.8^2} = 1$




Q5) Prove the uniqueness of one

I choosed a)

a) $A = LU$ where $A$ is non singular, $L$ lower triangular with $l_{ii} = 1$ and $U$ is upper triangular.

Let say that $A$ is a non singular $n \times n$ matrix.
Suppose $A = L_1U_1 = L_2U_2$ where
$L_1, L_2$ are lower triangular $n\times n$ matrixes and
$U_1, U_2$ are upper triangular $n\times n$ matrices.
(Notice that the determinant of a triangular matrice is equal to the product of a elements on the diagonal)
$\det A = \det(L_iU_i) = \det L_i \det U_i \neq 0$ for $i=1,2$ (A is non singular)
$\Rightarrow \det L_i \neq 0$ and $\det U_i \neq 0$ for $i=1,2$.

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$\Rightarrow L_i, U_i$ have inverses. for $i=1,2$

$$L_1 U_1 = L_2 U_2$$

$$L_2^{-1} L_1 U_1 = L_2^{-1} L_2 U_2$$

$$L_2^{-1} L_1 U_1 U_1^{-1} = L_2^{-1} L_2 U_2 U_1^{-1} \Leftrightarrow \underbrace{L_2^{-1} L_1}_{\text{lower triangular}} = \underbrace{U_2 U_1^{-1}}_{\text{upper triangular}}$$

$L_2^{-1}, L_1$ are lower triangular $\Rightarrow$ product $L_2^{-1} L_1$ is lower triangular

$U_2 U_1^{-1}$ are upper triangular $\Rightarrow$ product $U_2 U_1^{-1}$ is upper triangular

$$L_2^{-1} L_1 = U_2 U_1^{-1}$$

lower triangular = upper triangular $\Rightarrow$ diagonal

$$(L_2^{-1} L_1)_{ii} = (L_2^{-1})_{ii} (L_1)_{ii} = \frac{1}{1} \cdot 1$$

because $l_{ii}^{(1)} = 1 = (L_1)_{ii}$

$\frac{1}{l_{ii}^{(2)}} = \frac{1}{1} = (L_2^{-1})_{ii}$

hypo ($l_{ii} = 1$)

$$\Rightarrow (L_2^{-1} L_1)_{ii} = 1$$

$$\Rightarrow L_2^{-1} L_1 = I$$

$$\Rightarrow L_1 = L_2$$

Therefore $U_2 U_1^{-1} = I$

$$\Rightarrow U_1 = U_2$$

Therefore the LU decomposition is unique.

Q6)

a) $P^T = P$

$$P^T = \left(I - 2\frac{vv^T}{v^Tv}\right)^T = I^T - 2\frac{(vv^T)^T}{v^Tv} = I - 2\frac{v\,v^T}{v^Tv} = P$$

because $I^T = I$, $(vv^T)^T = vv^T$

b) P is orthogonal $P^{-1} = P^T$

$$PP = P^TP = \left(I - 2\frac{vv^T}{v^Tv}\right)\left(I - 2\frac{vv^T}{v^Tv}\right) = I - 4\frac{vv^T}{v^Tv} + 4\frac{vv^T}{v^Tv}\frac{vv^T}{v^Tv}$$

because $v^Tv = [v_1 \dots v_n]\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}$

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$$= I - \frac{4vv^T}{v^Tv} + \frac{4v(v^Tv)v^T}{(v^Tv)^2}$$

$$= I - \frac{4vv^T}{v^Tv} + \frac{4vv^T}{v^Tv} = I$$

$$\therefore \quad PP = P^TP = I$$

$$\Rightarrow PP = I \quad \Rightarrow P^{-1} = P$$

$$\Rightarrow P^{-1} = P \underset{\uparrow\,(a)}{=} P^T$$

c) Find the eigenvalues of P

we are looking for $\lambda$ such that

$(P - \lambda I)x_i = 0$ on cherche les x tel que

$Px_i = \lambda x_i$

$x_i \perp v, \quad i = 1, \dots, n-1$

$$Px_i = \left(I - \frac{2vv^T}{v^Tv}\right)x_i = Ix_i - \frac{2vv^Tx_i}{v^Tv}$$

$x_i \perp v$

$$= Ix_i - \frac{2v}{v^Tv}\underbrace{\cancel{v^Tx_i}}_{\to 0}$$

vecteur propre associé au $\lambda_1 \dots \lambda_{n-1}$

$$= Ix_i$$

$-1$

$$\therefore \lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = 1$$

but

$$Pv = \left(I - \frac{2vv^T}{v^Tv}\right)v = Iv - \frac{2vv^Tv}{v^Tv}$$

vect associé à $\lambda_n$

$$= v - \frac{2v(v^Tv)}{\cancel{v^Tv}}$$

$$= v - 2v$$

$$= -v$$

$$\Rightarrow \lambda_n = -1$$

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d) because $P$ is orthogonal, symmetric and is own inverse

$$\det P = \det\left( I - \frac{2vv^T}{v^Tv} \right)$$

$$= \lambda_1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_{n-1} \cdot \lambda_n$$

$$= 1 \cdot 1 \cdot 1 \cdot \cdots \cdot 1 \cdot -1$$

$$= -1$$