MAT 3380 SOLUTION 1st set (S1) T1/2

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1. a) $x_{appr} = 0.257$ since in the last 3 iteration these first three decimal digits stay constant, whereas the 4^{th} decimal digits varies.

b) Order of convergence

FPI: 1

Aitken: 1.3

Steffensen: 2, since corresponding values of $r_n(q)$ tend to constants (−0.259, 0.0299, 0.0, respectively).

c) $g(x) = \frac{1}{3}(2 - e^x + x^2)$

$g'(x) = \frac{1}{3}(-e^x + 2x)$

$g'(0.2575) = \frac{1}{3}(-e^{0.2575} + 2\times 0.2575) = -0.2596 \neq 0$

Since $g'(0.2575) \neq 0$, FPI converges to first order.

(2) a) We can see that the modified Newton's method with $m=2$ converges the fastest to $p = 1.000$. That is, from the table of $r_n(k)$, we see that $r_n(1)$ and $r_n(3)$ vary, whereas only $r_n(2)$ appraches to the constant (0.3330). So, we conclude that the zero of $f(x)$ at $p=1.0$ is of order 2.

b) $f(x) = (x-1) - \ln(x)$

$f(1) = (1-1) - \ln(1) = 0 - 0 = 0$

$f'(x) = 1 - \frac{1}{x} \Rightarrow f'(1) = 1 - \frac{1}{1} = 0$

$f''(x) = +\frac{1}{x^2} \Rightarrow f''(1) = \frac{1}{1} = 1 \neq 0$

$\Rightarrow f(x)$ has zero of order 2 at $x = 1$.

2c) Suppose that function $h(x)$ has a simple zero at $x=p$.
Then $h(x)=(x-p)\cdot q(x)$, where $q(p)\neq 0$.
By Newton's method,

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}$$

So consider $g(x) = x - \frac{h(x)}{h'(x)}$.

$$g'(x) = 1 - \frac{(h'(x))^2 - h(x)h''(x)}{(h'(x))^2} = 1 - 1 + \frac{h(x)h''(x)}{(h'(x))^2} = \frac{h(x)h''(x)}{(h'(x))^2}$$

$h(p)=0 \Rightarrow g'(p)=0$

Then, $h'(x) = q(x) + (x-p)\,q'(x)$

$$h''(x) = q'(x) + q'(x) + (x-p)q''(x) = 2q'(x) + (x-p)\,q''(x)$$

$$g'(x) = \frac{(x-p)\,q(x)\cdot[2q'(x)+(x-p)\,q''(x)]}{[q(x)+(x-p)\,q'(x)]^2}$$

$$g'(p) = \frac{0\cdot q(p)[2q'(p)+0]}{[q(p)+0\cdot q'(x)]^2} = \frac{0}{q^2(p)} = 0$$ ($q(p)\neq 0$, so division is defined)

Since $g'(p)=0$, Newton's method converges to second order.

(3) $f(x) = x^2 - x^2 e^{-x} + \frac{x^4}{24}$

$f'(x) = 2x - 2xe^{-x} + x^2 e^{-x} - \frac{x^3}{6}$

$f''(x) = 2 - 2e^{-x} + 2xe^{-x} + 2xe^{-x} - x^2e^{-x} - \frac{x^2}{2}$

$f''(x) = 2 - (2+4x+x^2)\,e^{-x} - \frac{x^2}{2}$

From the graph of $f''(x)$, we can see that $f''(x)$ is decreasing on $x\in[1,2]$, so

$$M = \max_{x\in[1,2]} |f''(x)| = |f''(1)| = 2-(2-4\times1+1)e^{-1} - \frac{1^2}{2} = 1.867879$$

$$|Err| = \left|\frac{(b-a)h^2 f''(\xi)}{12}\right| \leq \frac{(2-1)h^2}{12} M \leq 10^{-3}$$

$$h^2 \leq \frac{12\times10^{-3}}{1.867879} = 0.0064244$$

$$h \leq 0.080152$$

(51) T1. 3/3

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$nh = b-a \Rightarrow n = \frac{b-a}{h} \Rightarrow n \geq \frac{2-1}{0.080152} = \frac{1}{0.080152} = 12.476$

$\Rightarrow \boxed{n = 13}$

$\boxed{h = \frac{b-a}{n} = \frac{2-1}{13} = 0.076923}$ ✓

(4) $\int_0^1 x^2 e^{-x}\,dx$

$f(x) = x^2 e^{-x}$

$h = \frac{1}{2}$, $R_{2,1} = \frac{h}{2}\left[f(0) + 2f(0.5) + f(1)\right]$

a) $R_{2,1} = \frac{1}{2\times 2}\left[0^2 e^{-0} + 2(0.5)^2 e^{-0.5} + 1^2 e^{-1}\right] = \underline{0.16778619}$

$R_{4,2} = R_{4,1} + \frac{R_{4,1} - R_{3,1}}{4^{2-1} - 1} = 0.16107990 + \frac{0.16107990 - 0.16248891}{3}$

b) $\boxed{R_{4,2} = 0.16061040}$

(5) a) $f[x_1, x_2] = \frac{f[x_1] - f[x_2]}{x_1 - x_2} = \frac{-5.89483 - (-5.65014)}{0.1 - 0.3} = \underline{1.22345}$

b) $f[x_2, x_3] = \frac{f[x_2] - f[x_3]}{x_2 - x_3} = \frac{-5.65014 - (-5.17788)}{0.3 - 0.6} = \underline{1.57420}$

c) $p_3(x) = -6.0 + x \cdot 1.05170 + x(x-0.1)\,0.57250 +$
$+ x(x-0.1)(x-0.3) \cdot 0.21500$

d) $p_3(0.2) = -6.0 + 0.2 \times 1.05170 + 0.2(0.2-0.1) \cdot 0.57250 +$
$+ 0.2(0.2-0.1)(0.2-0.3) \cdot 0.21500 = \underline{-5.77864}$ ✓