Test mi-session 1 | **MAT 3380** | **Mid-term 1**

Durée: 80 min
Place: MRT219
17 février 2006
10:00–11:20

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Prof.: Rémi Vaillancourt

Instructions:

(a) *Test à livre fermé. Tout genre de calculatrices autorisé.*
Closed book mid-term. All types of calculators are allowed.
(b) *Répondre sur le questionnaire.*
Answer on the question sheets.
(c) *Les 6 questions ont toutes la même valeur.*
All six questions have the same value.
(d) *Une réponse sans calcul à l'appui ne sera pas corrigée.*
Show all computation. Bare answers will not be graded.
(e) *Tous les angles sont en RADIANS.*
Prière de tester et d'ajuster votre calculatrice.
All angles are in RADIAN measures.
Test and adjust your calculators.

$$\sin 1.123456789 = 0.90160112364453$$

"The purpose of computing is insight, not numbers", Hamming

Carte d'ét. : Oui Non

Stud. Card : Yes No

Qu. 1. Consider the fixed point iteration $x_{n+1} = g(x_n)$:

$$x_{n+1} = \frac{1}{3}\left(2 - e^{x_n} + x_n^2\right), \qquad x_1 = 0.5.$$

Numerical iterates obtained by this fixed point iteration, Aitken's acceleration and Steffensen's acceleration are listed in the following table:

	FPI	Aitken	Steffensen
x1	0.5000	0.2587	0.5000
x2	0.2004	0.2576	0.2587
x3	0.2727	0.2575	0.2575
x4	0.2536	0.2575	0.2575
x5	0.2586	0.2575	0.2575
x6	0.2573	0.2575	NaN
x7	0.2576	0.2575	NaN
x8	0.2575	0.2575	NaN

Qu. 1(a): Write down an approximate value $x_{\mathbf{appr}}$ of the fixed point and explain your answer:

$$x_{\text{appr}} =$$

Let

$$\varepsilon_n = y_{n+1} - y_n \quad \text{and} \quad r_n(q) = \varepsilon_{n+1}/\varepsilon_n^q.$$

The results for the fixed point iteration x_n, Aitken's acceleration a_n and Steffensen's iteration s_n are, respectively:

$r_n(1)$	$r_n(1.3)$	$r_n(2)$
-0.24141908405732	0.03283711358292	0.00002287302451
-0.26467314169051	0.02968834769036	0.00000000078230
-0.25824069094927	0.03009078416098	0.00000000000000
-0.25989942037801	0.02996773583712	NaN
-0.25946794316388	0.02999829638646	NaN

Qu. 1(b): From this table write down the order of convergence of the three methods:

FPI: ; Aitken: ; Steffensen:

Qu. 1(c): Find analytically the order of the fixed point iteration by evaluating the derivative $g'(0.2575)$.

Qu. 2. Suppose $f(x)$ has a zero of multiplicity $m \geq 1$ at $x = a$. Then it is known that the modified Newton method

$$x_{n+1} = x_n - m\,\frac{f(x_n)}{f'(x_n)}$$

converges to second order, with starting value x_0 sufficiently near a.

Newton's and the modified Newton methods with $m = 1, 2, 3$ applied to the equation

$$f(x) = (x - 1) - \ln x$$

produce the following results:

m=1	m=2	m=3
1.5000	1.5000	1.5000
1.2164	0.9328	0.6492
1.1011	0.9984	1.1001
1.0489	1.0000	0.9452
1.0241	1.0000	1.0259
1.0119	1.0000	0.9867
1.0059	1.0000	1.0065

With $r_n(k)$ as defined in question 1 the above three methods produce the following table:

$r_n(1)$	$r_n(2)$	$r_n(3)$
0.4064	0.2041	-0.5300
0.4529	0.3614	-0.3436
0.4764	0.3339	-0.5208
0.4882	0.3330	-0.4849

Qu. 2(a): From the last table, which method converges fastest to the root $p = 1.000$ and deduce the order q of the zero of the function $f(x)$ at p.

Qu. 2(b): Show analytically that indeed $f(x)$ has a zero of order q at $x = 1$.

Qu. 2(c): Show analytically that Newton's method ($m = 1$) converges to second order at a simple zero of $h(x) = 0$.

Qu. 3. Consider the composite trapezoidal integration formula

$$\int_a^b f(x)\,dx = \frac{h}{2}\big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\big] - \frac{(b-a)h^2}{12} f''(\xi), \qquad a < \xi < b.$$

Determine the values of h and n to approximate

$$\int_0^2 \left(x^2 - x^2\,e^{-x} + \frac{x^4}{24}\right) dx$$

to 10^{-3} by the composite trapezoidal rule.

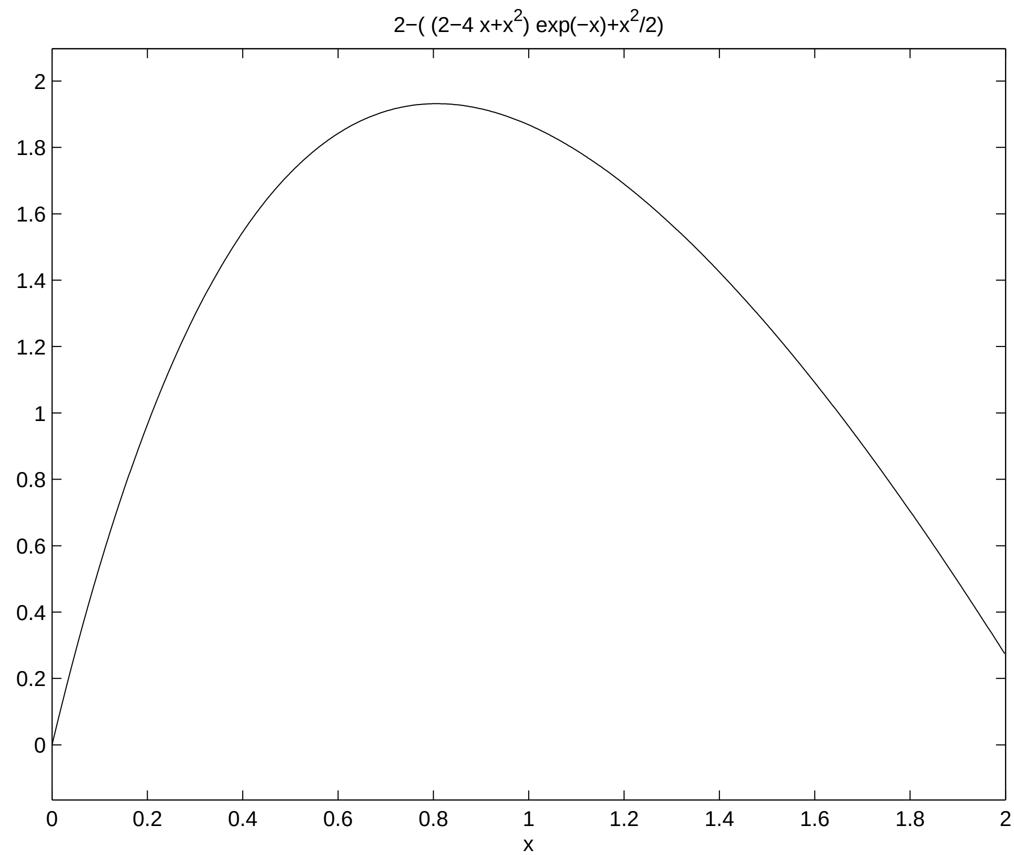

Figure 1. The graph of $f''(x)$ on $[0, 2]$ for question 3.

Qu. 4. The integral

$$\int_0^1 x^2 \, e^{-x} \, dx$$

is evaluated by the composite trapezoidal rule (see Qu. 3) with

$$h_1 = 1, \quad h_2 = \frac{1}{2}, \quad \ldots, \quad h_6 = \frac{1}{2^5},$$

and produces the first column ($j = 1$) of the following table with $R_{2,1}$ and $R_{4,3}$ removed:

	j=1	j=2	j=3	j=4	j=5	j=6
h1 k=1	0.18393972					
h2 k=2		0.16240168				
h3 k=3	0.16248841	0.16072248	0.16061053			
h4 k=4	0.16107990		0.16060292	0.16060280		
h5 k=5	0.16072243	0.16060327	0.16060280	0.16060279	0.16060279	
h6 k=6	0.16063272	0.16060282	0.16060279	0.16060279	0.16060279	0.16060279

The remaining columns are produced by Romberg's integration scheme:

$$R_{k,j} = R_{k,j-1} + \frac{R_{k,j-1} - R_{k-1,j-1}}{4^{j-1} - 1}.$$

Qu. 4(a): Compute $R_{2,1}$ by the composite trapezoidal rule with $h = 1/2$.

Qu. 4(b): Compute $R_{4,2}$.

Qu. 5. Consider the following incomplete divided difference table:

TABLE 1. Divided difference table for Question 5.

x	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, \ldots, x_{i+3}]$	$f[x_i, \ldots, x_{i+4}]$
0.0	-6.00000				
		1.05170			
0.1	-5.89483		0.57250		
		$f[x_1, x_2]$		0.21500	
0.3	-5.65014		$f[x_1, x_2, x_3]$		0.06301
		$f[x_2, x_3]$		$f[x_1, x_2, x_3, x_4]$	
0.6	-5.17788		$f[x_2, x_3, x_4]$		
		$f[x_3, x_4]$		$f[x_2, x_3, x_4, x_5]$	
1.0	-4.28172		$f[x_3, x_4, x_5]$		
		$f[x_4, x_5]$			
1.1	-3.99583				

The forward divided difference Gregory–Newton interpolation polynomial with $x_0 = 0$ is

$$p_n(x) = f_0 + (x - x_0)\, f[x_0, x_1] + (x - x_0)\,(x - x_1)\, f[x_0, x_1, x_2] + \cdots \\ + (x - x_0)\,(x - x_1) \cdots (x - x_{n-1})\, f[x_0, x_1, \ldots, x_n],$$

Qu. 5(a): Compute $f[x_1, x_2]$.

Qu. 5(b): Compute $f[x_2, x_3]$.

Qu. 5(c): Write down the Gregory-Newton forward difference polynomial $p_3(x)$ with $x_0 = 0$ for the data in the above difference table.

Qu. 5(d): Interpolate the data $\{f[x_i]\}$ at $x = 0.2$ by means of $p_3(x)$.