Mat 3380 - Assignment 7

pg 166-167 - #5.2, 5.7, 5.12, 5.18, 5.21 (use starting values fro Ex 5.12). In each case, take $h=0.1$ and do 5 steps up to $x=0.$ Plot the solution only for Ex. 12.

#5.2) Use Euler's method with $h=0.1$ to obtain a four decima approximation for each initial value problem on $0\le x\le 1$ and plot the numerical solution.

$$y' = x + \sin y, \quad y(0) = 0$$

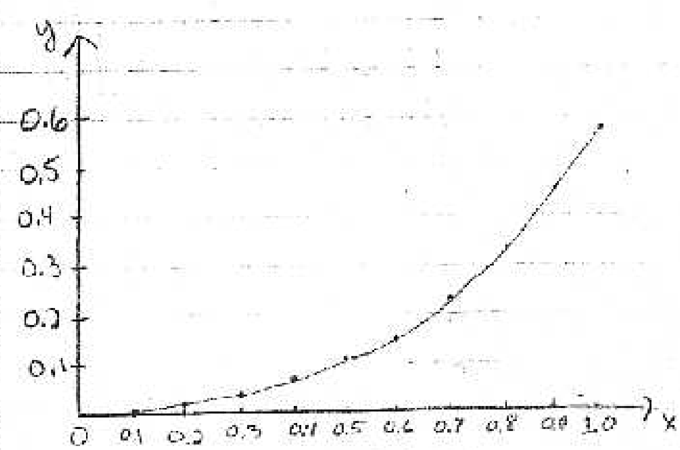
$$x_0 = 0 \qquad y_0 = 0 \qquad f(x,y) = x + \sin y \qquad x_n = x_0 + 0.1n = 0.1n$$

$$N = \frac{1-0}{0.1} = 10$$

Euler's equation : $y_{n+1} = y_n + 0.1\, f(0.1n, y_n)$

$n=0$: $y_1 = y_0 + 0.1 f(0,0) = 0,0000$
$n=1$; $y_2 = y_1 + 0.1 f(0.1, 0) = 0.1(0.1 + \sin 0) = 0.0100$
$n=2$; $y_3 = y_2 + 0.1 f(0.2, 0.01) = 0.01 + 0.1(0.2 + \sin(0.01))$
$= 0.0310$
⋮

n	x_n	y_n
0	0	0.0000
1	0.1	0.0000
2	0.2	0,0100
3	0.3	0,0310
4	0.4	0,0641
5	0.5	0,1105
6	0.6	0,1715
7	0.7	0,2486
8	0.8	0,3437
9	0.9	0,4569
10	1.0	0.5910



#5.7) Use the improved Euler method with $h=0.1$ to obtain a four-decimal approximation for each initial value problem on $0 \le x \le 1$ and plot the numerical solution

$$y' = x + \sin y \qquad y(0) = 0$$

$$x_0 = 0, \; y_0 = 0 \qquad f(x,y) = x + \sin y \qquad N = \frac{1-0}{0.1} = 10$$

$$x_n = x_0 + 0.1n = 0.1n$$

$$y_{n+1}^{P} = y_n^{c} + 0.1\, f(0.1n, y_n^{c}) \quad \text{and}$$

$$y_{n+1}^{c} = y_n^{c} + 0.05\left[f(0.1n, y_n^{c}) + f(0.1(n+1), y_{n+1}^{P}) \right]$$

n	x_n	y_n^{P}	y_n^{c}
0	0	0.0000	0.0000
1	0.1	0.0000	0.0050
2	0.2	0.0155	0.0210
3	0.3	0.0431	0.0492
4	0.4	0.0841	0.0909
5	0.5	0.1400	0.1474
6	0.6	0.2121	0.2203
7	0.7	0.3021	0.3111
8	0.8	0.4117	0.4214
9	0.9	0.5423	0.5527
10	1.0	0.6952	0.7060

$n=0$: $y_1^{P} = y_0 + 0.1(0 + \sin(0))$
$= 0$

$y_1^{c} = y_0 + 0.05[(0+\sin(0)) + (0.1 + \sin($
$= 0.005$

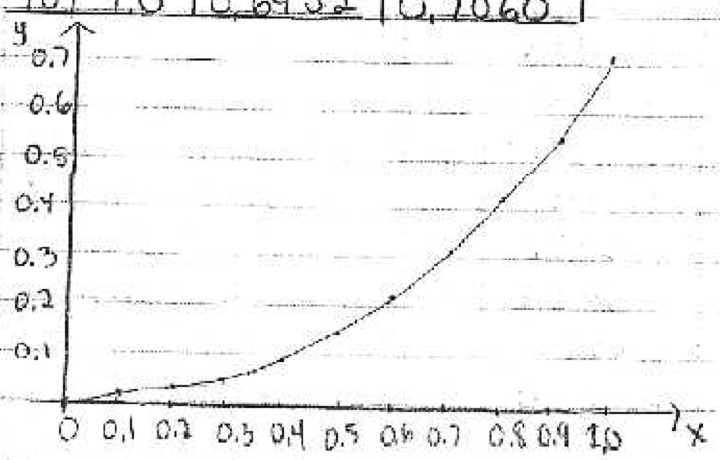
$n=1$: $y_2^{P} = y_1^{c} + 0.1(0.1 + \sin(0.005))$
$= 0.01549999$

$y_2^{c} = y_1^{c} + 0.05((0.1 + \sin(0.005)) + (0.2 + \sin$
$= 0.02102$

$n=2$: $y_3^{P} = y_2^{c} + 0.1(0.2 + \sin(0.02102))$
$= 0.043122$

$y_3^{c} = y_2^{c} + 0.05[(0.2 + \sin(0.02102)) + (0.3 + \sin($
$= 0.0492$

⋮



#5.12) Use the Runge-Kutta method of order 4 with $h=0.1$ to obtain a six-decimal approximation for each initial value problem on $0 \le x \le 1$ and plot the numerical solution.

$y' = x + \sin y \qquad y(0) = 0 \qquad h = 0.1$ do 5 steps

$N = 10 \qquad x_n = 0.1n$

$$k_1 = 0.1f(0.1n, y_n) = 0.1(0.1n + \sin(y_n))$$

$$k_2 = 0.1f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1\right) = 0.1f\left(0.1n + 0.05, y_n + \tfrac{1}{2}k_1\right)$$

$$k_3 = 0.1f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2\right) = 0.1f\left(0.1n + 0.05, y_n + \tfrac{1}{2}k_2\right)$$

$$k_4 = 0.1f\left(0.1n + 0.1, y_n + k_3\right) = 0.1f\left(0.1(n+1), y_n + k_3\right)$$

$$y_{n+1} = y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

n	x	y	k_1	k_2	k_3	k_4
0.000000	0.000000	0.000000	0.000000	0.005000	0.005250	0.010525
1.000000	0.100000	0.005171	0.010517	0.016043	0.016319	0.022149
2.000000	0.200000	0.021403	0.022140	0.028247	0.028552	0.034993
3.000000	0.300000	0.049858	0.034984	0.041730	0.042066	0.0491795
4.000000	0.400000	0.091817	0.049169	0.056614	0.056984	0.064825
5.000000	0.500000	0.148682	0.064813	0.073010	0.073413	0.082027

$n = 0 \qquad k_1 = 0.1(0 + \sin(0)) = 0$

$k_2 = 0.1f(0 + 0.05, 0) = 0.1(0.05) = 0.005$

$k_3 = 0.1f(0.05, 0 + \tfrac{1}{2}(0.005)) = 0.00525$

$k_4 = 0.1f(0.1(1), 0 + 0.00525)$

$= 0.1(0.1 + \sin(0.00525)) = 0.010525$

$y_1 = y_0 + \tfrac{1}{6}(0 + 2(0.005) + 2(0.00525) + 0.010525)$

$= 0.005171$

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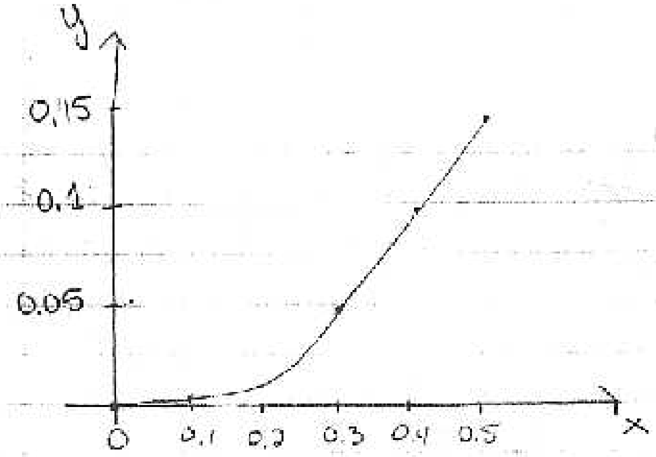

#5.18) Use the MatLab ode 23 embedded pair of order 3 with $h=0.1$ to obtain a six-decimal approximation for each initial value problem on $0 \le x \le 1$ and estimate the local truncation error by means of the given formula.

$k_1^{[n+1]} = k_4^{[n]}$

$$y' = x + 2\cos y \qquad y(0) = 0$$

$x_n = 0.1n$

$x_0 = 0 \quad y_0 = 0$

$$k_1 = 0.1 f(x_n, y_n)$$
$$k_2 = 0.1 f(x_n + 0.05, y_n + \tfrac{1}{2} k_1)$$
$$k_3 = 0.1 f(x_n + 0.075, y_n + \tfrac{3}{4} k_2)$$

$$k_4 = 0.1 f\left(x_n + 0.1, y_n + \tfrac{2}{9} k_1 + \tfrac{1}{3} k_2 + \tfrac{4}{9} k_3\right)$$

$$y_{n+1} = y_n + \tfrac{2}{9} k_1 + \tfrac{1}{3} k_2 + \tfrac{4}{9} k_3$$

$$E = -\tfrac{5}{72} k_1 + \tfrac{1}{12} k_2 + \tfrac{1}{9} k_3 - \tfrac{1}{8} k_4$$

$n=0$: $k_1 = 0.1 f(x_0, y_0) = 0.1(0 + 2\cos(0)) = 0.2$

$k_2 = 0.1 f(0 + 0.05, 0 + \tfrac{1}{2}(0.2)) = 0.204001$

$k_3 = 0.1 f(0 + 0.075, 0 + \tfrac{3}{4}(0.204001)) = 0.205164$

$k_4 = 0.1 f\left(0.1, 0 + \tfrac{2}{9}(0.2) + \tfrac{1}{3}(0.204001) + \tfrac{4}{9}(0.205164)\right)$

$= 0.1 f(0.1, 0.203629) = 0.205868$

$$y_1 = y_0 + \tfrac{2}{9} k_1 + \tfrac{1}{3} k_2 + \tfrac{4}{9} k_3 = 0.203629$$

$$E_1 = \tfrac{-5}{72}(0.2) + \tfrac{1}{12}(0.204001) + \tfrac{1}{9}(0.205164) - \tfrac{1}{8}(0.205868) = 1.737 \times 10^{-4}$$

n	x	y	k_1	k_2	k_3	k_4	E
0	0.0	0.000000	0.200000	0.204001	0.205164	0.205868	1.737×10^{-4}
1	0.1	0.203629	0.205868	0.205675	0.204828	0.203506	1.64×10^{-4}
2	0.2	0.408970	0.203506	0.199478	0.197102	0.194126	1.253×10^{-4}
3	0.3	0.608287	0.194126	0.187277	0.184009	0.179966	7.508×10^{-5}
4	0.4	0.795634	0.179966	0.171562	0.167978	0.163477	2.879×10^{-5}
5	0.5	0.967471	0.163477	0.154651	0.151155	0.146681	-5.111×10^{-6}

therefore an estimate local truncation error is -5.111×10^{-6}

#4.3) Solve the following system by the LU decomposition with partial pivoting.

$$2x_1 - x_2 + 5x_3 = 4$$
$$-6x_1 + 3x_2 - 9x_3 = -6$$
$$4x_1 - 3x_2 = -2$$

$$A = \begin{bmatrix} 2 & -1 & 5 \\ -6 & 3 & -9 \\ 4 & -3 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$|-6| > |4|$ and $|-6| > |2|$

therefore lets find P_1 such that it interchange the 1st and 2nd row.

$$P_1A = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P_1} \begin{bmatrix} 2 & -1 & 5 \\ -6 & 3 & -9 \\ 4 & -3 & 0 \end{bmatrix} = \begin{bmatrix} -6 & 3 & -9 \\ 2 & -1 & 5 \\ 4 & -3 & 0 \end{bmatrix}$$

Now, lets find M_1 such that M_1P_1A as zeros under -6.

$$M_1P_1A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 0 & 1 \end{bmatrix}}_{M_1} \begin{bmatrix} -6 & 3 & -9 \\ 2 & -1 & 5 \\ 4 & -3 & 0 \end{bmatrix} = \begin{bmatrix} -6 & 3 & -9 \\ 0 & 0 & 2 \\ 0 & -1 & -6 \end{bmatrix} = A_1$$

$|-1| > |0|$ therefore let's find P_2 such that it interchange the 2^{nd} and last row.

$$P_2 A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -6 & 3 & -9 \\ 0 & 0 & 2 \\ 0 & -1 & -6 \end{bmatrix} = \begin{bmatrix} -6 & 3 & -9 \\ 0 & -1 & -6 \\ 0 & 0 & 2 \end{bmatrix}$$

We can see that it already is upper triangular therefore

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$M_2 P_2 A_1 = \begin{bmatrix} -6 & 3 & -9 \\ 0 & -1 & -6 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$M_2 P_2 M_1 P_1 A = U$$

$$\Rightarrow \quad A = P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} U = LU$$

$$P_1^{-1} = P_1^T \text{ and } P_2^{-1} = P_2^T$$

$$P_1^T = P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_2^T = P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 0 & 1 \end{bmatrix} \Rightarrow M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -2/3 & 0 & 1 \end{bmatrix} \text{ and } M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -2/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 0 & 1 \\ -2/3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1/3 & 0 & 1 \\ 1 & 0 & 0 \\ -2/3 & 1 & 0 \end{bmatrix}$$

Now: $Ly = b$

$$\begin{bmatrix} -1/3 & 0 & 1 \\ 1 & 0 & 0 \\ -2/3 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix}$$

$y_1 = -6$

$-\frac{2}{3} y_1 + y_2 = -2$

$\Rightarrow y_2 = -2 + \frac{2}{3}(-6) = -6$

$-\frac{1}{3} y_1 + y_3 = 4$

$\Rightarrow y_3 = 4 + \frac{(-6)}{3} = 2$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -6 \\ -6 \\ 2 \end{bmatrix}$$

$Ux = y$

$$\begin{bmatrix} -6 & 3 & -9 \\ 0 & -1 & -6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ -6 \\ 2 \end{bmatrix}$$

$\Rightarrow 2x_3 = 2 \Rightarrow x_3 = 1$

$-x_2 - 6x_3 = -6$

$\Rightarrow -x_2 = -6 + 6x_3$

$\Rightarrow x_2 = 6 - 6(1) = 0$

$-6x_1 + 3x_2 - 9x_3 = -6$

$\Rightarrow x_1 = \frac{-6 + 9(1)}{-6} = -\frac{1}{2}$

Therefore

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$$