MAT 3380 D6.1

SOLUTIONS

Ass. 6

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06.03.10

4.16. $f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

x_i	-1	-0.5	0	0.25	0.5	0.75	1
y_i	e^{-1}	$e^{-1/2}$	1	$e^{1/4}$	$e^{1/2}$	$e^{3/4}$	e

$$A^T A \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = A^T \vec{y}, \text{ where}$$

$$A = \begin{bmatrix} P_0(x_1) & P_1(x_1) & P_2(x_1) \\ \vdots & & \\ P_0(x_7) & P_1(x_7) & P_2(x_7) \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -0.5 & -0.125 \\ 1 & 0 & -0.5 \\ 1 & 0.25 & -0.40625 \\ 1 & 0.5 & -0.125 \\ 1 & 0.75 & 0.34375 \\ 1 & 1 & 1 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} e^{-1} \\ e^{-1/2} \\ 1 \\ e^{1/4} \\ e^{1/2} \\ e^{3/4} \\ e \end{bmatrix}$$

$$A^T A \vec{a} = \begin{bmatrix} 7 & 1 & 1.1875 \\ 1 & 3.125 & 0.15625 \\ 1.1875 & 0.15625 & 2.56445 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 9.74244 \\ 4.78025 \\ 2.57034 \end{bmatrix}$$

Using the Cholesky decomposition

$$A = \begin{bmatrix} 7 & 1 & 1.1875 \\ 1 & 3.125 & 0.15625 \\ 1.1875 & 0.15625 & 2.56445 \end{bmatrix} = \begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} g_{11} & g_{21} & g_{31} \\ 0 & g_{22} & g_{32} \\ 0 & 0 & g_{33} \end{bmatrix}$$

$g_{11}^2 = 7 \Rightarrow g_{11} = \sqrt{7} = 2.64575$

$g_{11} g_{21} = 1 \Rightarrow g_{21} = 1 \div 2.64575 = 0.377964$

$g_{11} g_{31} = 1.1875 \Rightarrow g_{31} = 1.1875 \div 2.64575 = 0.448833$

$g_{21}^2 + g_{22}^2 = 3.125 \Rightarrow g_{22}^2 = 2.98214 \Rightarrow g_{22} = 1.72689$

$g_{21} g_{31} + g_{22} g_{32} = 0.15625 \Rightarrow g_{32} = -0.0077554$

$g_{31}^2 + g_{32}^2 + g_{33}^2 = 2.56445 \Rightarrow g_{33} = 1.53719$

$$G = \begin{bmatrix} 2.64575 & 0 & 0 \\ 0.377964 & 1.72689 & 0 \\ 0.448833 & -0.0077554 & 1.53719 \end{bmatrix}$$

$G G^T \vec{a} = A^T \vec{y} \Rightarrow G^T \vec{a} = \vec{b} \quad G \vec{b} = A^T \vec{y}$

By forward substitution in $G\vec{b} = A^T \vec{y}$:

$\Rightarrow b_0 = 9.74244 \div 2.64575 = 3.68230$

$\Rightarrow b_1 = \frac{1}{1.72689}(4.78025 - 0.377964 \times 3.68230) = 1.96218$

$\Rightarrow b_2 = \frac{1}{1.53719}(2.51034 + 0.0077554 \times 1.96218 - 0.448833 \times 3.68230)$

$b_2 = 0.567802$

Then by backward substitution:

$\Rightarrow a_2 = 0.567802 \div 1.53719 = 0.369377$

$\Rightarrow a_1 = \frac{1}{1.72689}(1.96218 + 0.0077554 \times 0.369377) = 1.13791$

$\Rightarrow a_0 = \frac{1}{2.64575}(3.68230 - 0.377964 \times 1.13791 - 0.448833 \times 0.369377)$

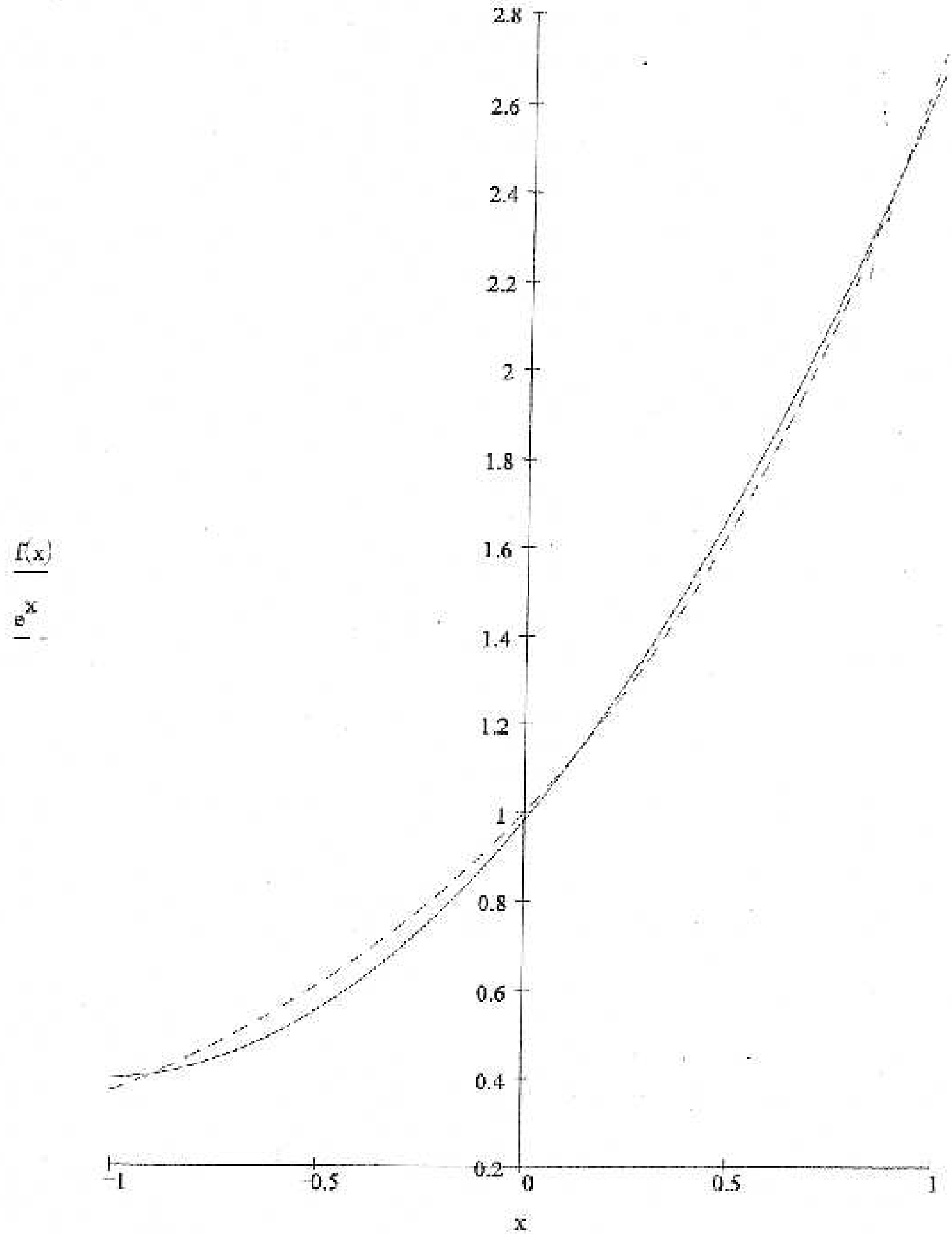
$a_0 = 1.16656$

$$\vec{a} = \begin{bmatrix} 1.16656 \\ 1.13791 \\ 0.369377 \end{bmatrix}$$

$f(x) = 1.16656 + 1.13791\, x + 0.369377 \cdot \frac{1}{2}(3x^2 - 1)$

D6.3

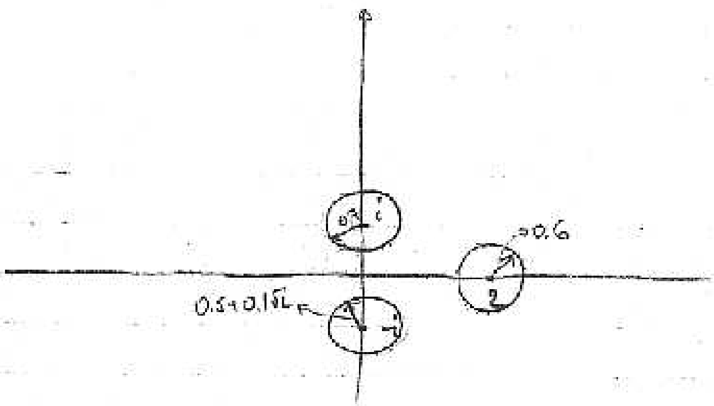
$f(x) := 1.16656 + 1.13791 \cdot x + 0.369377 \cdot 0.5 \cdot (3 \cdot x \cdot x - 1)$



4.17.

$$A = \begin{bmatrix} -i & 0.1+0.1i & 0.5i \\ 0.3i & 2 & 0.3 \\ 0.2 & 0.3+0.4i & i \end{bmatrix}$$

Center	radius
$c_1 = -i$	$r_1 = \lvert 0.1+0.1i \rvert + \lvert 0.5i \rvert = 0.1\sqrt{2}+0.5 = 0.641421$
$c_2 = 2$	$r_2 = \lvert 0.3i \rvert + \lvert 0.3 \rvert = 0.6$
$c_3 = i$	$r_3 = \lvert 0.2 \rvert + \lvert 0.3+0.4i \rvert = 0.7$

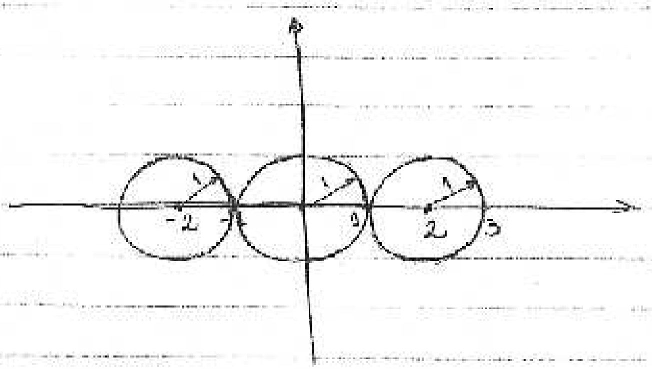


4.18.

$$B = \begin{bmatrix} -2 & 1/2 & i/2 \\ 1/2 & 0 & i/2 \\ -i/2 & -i/2 & 2 \end{bmatrix}$$

kot arkad !

Center	radius
$c_1 = -2$	$r_1 = \lvert \frac{1}{2} \rvert + \lvert \frac{i}{2} \rvert = 1$
$c_2 = 0$	$r_2 = \lvert \frac{1}{2} \rvert + \lvert \frac{i}{2} \rvert = 1$
$c_3 = 2$	$r_3 = \lvert -\frac{i}{2} \rvert + \lvert -\frac{i}{2} \rvert = 1$



D6,5

4.19. $l_1(A) = \max\{|-i| + |0.3i| + |0.2|,\ |0.1+0.1i| + |2| + |0.3+0.4i|,$
$|0.5i| + |0.3| + |i|\} = \max\{1.5,\ 2.5+0.1\sqrt{2},\ 1.8\} =$
$= 2.5 + 0.1\sqrt{2} \approx 2.641421$

$l_\infty(B) = \max\{|-2| + |\frac{1}{2}| + |\frac{i}{2}|,\ |\frac{1}{2}| + |0| + |\frac{i}{2}|,\ |-\frac{i}{2}| + |-\frac{i}{2}| + |2|\}$
$= \max\{3, 1, 3\} = 3$

4.21

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 2 & 6 & 6 \\ 3 & 6 & 3 \end{bmatrix} \qquad X^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$X^{(1)} = A X^{(0)} = \begin{bmatrix} 8 \\ 14 \\ 12 \end{bmatrix} \qquad u^{(1)} = \frac{X^{(1)}}{\|X^{(1)}\|_\infty} = \frac{X^{(1)}}{14} = \begin{bmatrix} 4/7 \\ 1 \\ 6/7 \end{bmatrix}$$

$$X^{(2)} = A u^{(1)} = \begin{bmatrix} 44/7 \\ 86/7 \\ 72/7 \end{bmatrix} \qquad u^{(2)} = \frac{X^{(2)}}{86} \cdot 7 = \begin{bmatrix} \frac{44}{86} \\ 1 \\ \frac{72}{86} \end{bmatrix} = \begin{bmatrix} \frac{22}{43} \\ 1 \\ \frac{36}{43} \end{bmatrix}$$

$$X^{(3)} = A u^{(2)} = \begin{bmatrix} 260/43 \\ 518/43 \\ 432/43 \end{bmatrix} \qquad u^{(3)} = \frac{X^{(3)}}{518} \cdot 43 = \begin{bmatrix} \frac{260}{518} \\ 1 \\ \frac{432}{518} \end{bmatrix} = \begin{bmatrix} 130/259 \\ 1 \\ 216/259 \end{bmatrix}$$

$$\lambda_1 \approx \frac{518}{43} \qquad X_1 \approx \begin{bmatrix} 130/259 \\ 1 \\ 216/259 \end{bmatrix} = \begin{bmatrix} 0.5019 \\ 1 \\ 0.8340 \end{bmatrix}$$

$= 12.0465$

✓ ✓

4.20. $A = \begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix}$

Eigenvalues

$$\begin{vmatrix} 10-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = 20 - 12\lambda + \lambda^2 - 16 = \lambda^2 - 12\lambda + 4 = 0$$

$$\lambda_{1,2} = \frac{12 \pm \sqrt{144-16}}{2}$$

$$\lambda_1 = 11.656854 \qquad \lambda_2 = 0.343146$$

Suppose we start with $\lambda = 11.60$, which is close to λ_1.

$$A - \lambda I = \begin{bmatrix} -1.6 & 4 \\ 4 & -9.6 \end{bmatrix} = A_1$$

$$P_1 A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1.6 & 4 \\ 4 & -9.6 \end{bmatrix} = \begin{bmatrix} 4 & -9.6 \\ -1.6 & 4 \end{bmatrix} = A_2$$

$$M_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0.4 & 1 \end{bmatrix} \begin{bmatrix} 4 & -9.6 \\ -1.6 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -9.6 \\ 0 & 0.16 \end{bmatrix} = U$$

$$M_1 P_1 A_1 = U \Rightarrow A_1 = \underbrace{P_1^{-1} M_1^{-1}}_{L} U$$

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -0.4 & 1 \end{bmatrix} = \begin{bmatrix} -0.4 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_1 = LU = \begin{bmatrix} -0.4 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -9.6 \\ 0 & 0.16 \end{bmatrix}$$

$LU y^{(1)} = x^{(0)}$. Put $U y^{(1)} = z^{(1)}$

$$\begin{bmatrix} -0.4 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} z_1 = 1 \\ -0.4 z_1 + z_2 = 1 \Rightarrow z_2 = 1.4 \end{array} \Bigg\} z = \begin{bmatrix} 1 \\ 1.4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -9.6 \\ 0 & 0.16 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.4 \end{bmatrix} \rightarrow \begin{array}{l} 0.16 y_2 = 1.4 \Rightarrow y_2 = 8.75 \\ 4y_1 - 9.6 y_2 = 1 \Rightarrow 4y_1 = 85 \Rightarrow y_1 = 21.25 \end{array}$$

$$y^{(1)} = \begin{bmatrix} 21.25 \\ 8.75 \end{bmatrix} \quad x^{(1)} = \frac{y^{(1)}}{\|y^{(1)}\|_\infty} = \frac{1}{21.25} y^{(1)} = \begin{bmatrix} 1 \\ 0.411765 \end{bmatrix} \checkmark$$

$LUy^{(2)} = x^{(1)}$. Put $Uy^{(2)} = z^{(2)}$.

$$\begin{bmatrix} -0.4 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.411765 \end{bmatrix} \Rightarrow z_1 = 0.411765$$

$$\Rightarrow -0.4z_1 + z_2 = 1 \Rightarrow z_2 = 1.164706$$

$$z^{(2)} = \begin{bmatrix} 0.411765 \\ 1.164706 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -9.6 \\ 0 & 0.16 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.411765 \\ 1.164706 \end{bmatrix} \Rightarrow 0.16\, y_2 = 1.164706$$

$$y_2 = 7.279412$$

$$4y_1 - 9.6y_2 = 0.411765$$

$$y_1 = 17.573529$$

$$y^{(2)} = \begin{bmatrix} 17.573529 \\ 7.279412 \end{bmatrix} \quad x^{(2)} = \frac{y^{(2)}}{17.573529} = \begin{bmatrix} 1 \\ 0.414226 \end{bmatrix} \checkmark$$

$LUy^{(3)} = x^{(2)}$. Put $Uy^{(3)} = z^{(3)}$

$$\begin{bmatrix} -0.4 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.414226 \end{bmatrix} \Rightarrow z_1 = 0.414226$$

$$-0.4z_1 + z_2 = 1 \Rightarrow$$

$$z_2 = 1.165690$$

$$z^{(3)} = \begin{bmatrix} 0.414226 \\ 1.165690 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -9.6 \\ 0 & 0.16 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0.414226 \\ 1.165690 \end{bmatrix} \Rightarrow 0.16\, y_2 = 1.165690$$

$$y_2 = 7.285565$$

$$4y_1 - 9.6y_2 = 0.414226$$

$$y_1 = 17.588912$$

$$y^{(3)} = \begin{bmatrix} 17.588912 \\ 7.285565 \end{bmatrix} \quad x^{(3)} = \frac{y^{(3)}}{17.588912} = \begin{bmatrix} 1 \\ 0.414214 \end{bmatrix}$$

? $\lambda_1 \simeq 11.60$ $\quad \vec{x}_1 \simeq \begin{bmatrix} 1 \\ 0.414214 \end{bmatrix} \checkmark$

$$Ax^{(3)} = \begin{bmatrix} 11.6569 \\ 4.8284 \end{bmatrix} \Rightarrow \lambda_1 \simeq 11.6569$$

(largest component of $Ax^{(3)}$)

Suppose we start with $\lambda = 0.30$, which is close to $\lambda_2 = 0.343146$.

$$A - \lambda I = \begin{bmatrix} 9.7 & 4 \\ 4 & 1.7 \end{bmatrix} = A_1$$

$$M_1 A_1 = \begin{bmatrix} 1 & 0 \\ -0.412371 & 1 \end{bmatrix} \begin{bmatrix} 9.7 & 4 \\ 4 & 1.7 \end{bmatrix} = \begin{bmatrix} 9.7 & 4 \\ 0 & 0.050516 \end{bmatrix} = U$$

$M_1 A = U \Rightarrow A = M_1^{-1} U$, $L = M_1^{-1}$

$$L = \begin{bmatrix} 1 & 0 \\ 0.412371 & 1 \end{bmatrix}$$

$$A_1 = LU = \begin{bmatrix} 1 & 0 \\ 0.412371 & 1 \end{bmatrix} \begin{bmatrix} 9.7 & 4 \\ 0 & 0.050516 \end{bmatrix}$$

$LU y^{(1)} = x^{(0)}$. Let $z^{(1)} = U y^{(1)}$.

$$\begin{bmatrix} 1 & 0 \\ 0.412371 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} z_1 = 1 \\ z_2 = 1 - 0.412371 z_1 = 0.587629 \end{array}$$

$$z^{(1)} = \begin{bmatrix} 1 \\ 0.587629 \end{bmatrix}$$

$$\begin{bmatrix} 9.7 & 4 \\ 0 & 0.050516 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.587629 \end{bmatrix}$$

$$\left.\begin{array}{l} \Rightarrow 0.050516\, y_2 = 0.587629 \Rightarrow y_2 = 11.632532 \\ 9.7 y_1 + 4 y_2 = 1 \Rightarrow y_1 = -4.693828 \end{array}\right\} \Rightarrow y^{(1)} = \begin{bmatrix} -4.693828 \\ 11.632532 \end{bmatrix}$$

$$x^{(1)} = \frac{y^{(1)}}{\|y^{(1)}\|_\infty} = \frac{y^{(1)}}{11.632532} = \begin{bmatrix} -0.403509 \\ 1 \end{bmatrix}$$

$LU y^{(2)} = x^{(1)}$. Put $U y^{(2)} = z^{(2)}$

$$\begin{bmatrix} 1 & 0 \\ 0.412371 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -0.403509 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} z_1 = -0.403509 \\ 0.412371 z_1 + z_2 = 1 \\ \Rightarrow z_2 = 1.166395 \end{array}$$

$$\begin{bmatrix} 9.7 & 4 \\ 0 & 0.050516 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -0.403509 \\ 1.166395 \end{bmatrix} = z^{(2)}$$

$0.050516\, y_2 = 1.166395 \Rightarrow y_2 = 23.087621$

$9.7 y_1 + 4 y_2 = -0.403509 \Rightarrow y_1 = -9.563092$

$$y^{(2)} = \begin{bmatrix} -9.563092 \\ 23.087621 \end{bmatrix} \Rightarrow x^{(2)} = \begin{bmatrix} -0.414173 \\ 1 \end{bmatrix}$$

$LUy^{(3)} = x^{(2)}$. Put $Uy^{(3)} = z^{(3)}$.

$$\begin{bmatrix} 1 & 0 \\ 0.412371 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -0.414173 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} z_1 = -0.414173 \\ z_2 = 1 + 0.414173 \times 0.412371 \\ z_2 = 1.170793 \end{array}$$

$$z^{(3)} = \begin{bmatrix} -0.414173 \\ 1.170793 \end{bmatrix}$$

$$\begin{bmatrix} 9.7 & 4 \\ 0 & 0.050516 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -0.414173 \\ 1.170793 \end{bmatrix}$$

$$0.050516\, y_2 = 1.170793 \Rightarrow y_2 = 23.176673$$

$$9.7 y_1 + 4 y_2 = -0.414173 \Rightarrow y_1 = -9.600087$$

$$y^{(3)} = \begin{bmatrix} -9.600089 \\ 23.176673 \end{bmatrix} \quad x^{(3)} = \begin{bmatrix} -0.414213 \\ 1 \end{bmatrix}$$

? $\lambda_2 \approx 0.30$ $\quad \vec{x}_2 \approx \begin{bmatrix} -0.414213 \\ 1 \end{bmatrix}$

$$Au^{(3)} = \begin{bmatrix} -0.1421 \\ 0.3431 \end{bmatrix} \Rightarrow \lambda_2 \approx 0.3431$$

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(*) LU decomposition is unique.

Let A be $n\times n$ matrix. Suppose that $A = L_1U_1 = L_2U_2$, where L_1, L_2 are $n\times n$ unit lower-triangular matrices and U_1, U_2 are $n\times n$ upper-triangular matrices. Notice that determinant of a triangular matrix equals the product of diagonal elements of that matrix. Then, $\det A = \det L_i \det U_i \neq 0$, $i=1,2$, so $\det L_i \neq 0$, $\det U_i \neq 0$, $i=1,2$ and these triangular matrices have non-zero diagonal elements.
Since $\det L_i \neq 0$, $\det U_i \neq 0$, $i=1,2$, they are invertible.

So, $L_1U_1 = L_2U_2 \quad / U_2^{-1}$

$L_1^{-1} / \quad L_1U_1U_2^{-1} = L_2$

$$U_1U_2^{-1} = L_1^{-1}L_2$$

Since product of upper (lower) triangular matrices is upper (lower) triangular matrix, it follows that the upper triangular matrix $U_1U_2^{-1}$ equals the lower triangular matrix $L_1^{-1}L_2$. Hence $U_1U_2^{-1} = L_1^{-1}L_2$ is a diagonal matrix.
Moreover, L_i, $i=1,2$, has units ~~at~~ on the diagonal, and then so has L_i^{-1}, $i=1,2$. Hence $L_1^{-1}L_2$ is a unit diagonal matrix, that is

$$L_1^{-1}L_2 = I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} = U_1U_2^{-1}.$$

Hence, $L_1 = L_2$, $U_1 = U_2$.