Nom / Name :

No d'ét. / Stud. No.:

MAT 2784 B

Test mi-session 2
Durée: 90 min
Place: SIM 224
18 mars 2009
17:30–19:00

Prof.: Rémi Vaillancourt

Midterm 2
Time: 90 min
Place: SIM 224
18th of March 2009
17:30–19:00

Instructions:

(a) *À livre fermé. Tout type de calculatrices permis.*
Closed book. Any type of calculators is allowed.

(b) *Répondre sur le questionnaire.*
Answer on the question sheets.

(c) *Les 6 questions sont d'égale valeur.*
All 6 questions have the same value.

(d) *Donner le détail de vos calculs.*
Show all computation.

(e) *Un formulaire sera distribué.*
Formulae will be distributed.

(f) *Tous les angles sont en RADIANS.*
Tester et ajuster votre calculatrice.
All angles are in RADIAN measures.
Test and adjust your calculators.

$$\sin 1.123456789 = 0.90160112364453$$

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
Total	/60

Qu. 1. *Trouver la solution générale.*

Find the general solution.

$$y''' + 3y'' - 4y' - 12y = 0.$$

On fait la substitution $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$, $y''' = \lambda^3 e^{\lambda x}$

le polynôme caractéristique est:

$$\lambda^3 + 3\lambda^2 - 4\lambda - 12 = 0$$

une racine évidente est $\lambda = -2$

$$\begin{array}{r|l} \lambda^3 + 3\lambda^2 - 4\lambda - 12 & (\lambda+2) \\ \cline{2-2} -\quad \lambda^3 + 2\lambda^2 \qquad\qquad & \lambda^2 + \lambda - 6 \\ \hline \lambda^2 - 4\lambda - 12 & \\ -\quad \lambda^2 + 2\lambda \qquad \\ \hline -6\lambda - 12 & \\ -\quad -6\lambda - 12 & \\ \hline 0 & \end{array}$$

$$\Rightarrow \lambda^3 + 3\lambda^2 - 4\lambda - 12 = (\lambda+2)(\lambda^2+\lambda-6) = 0$$

$$(\lambda+2)(\lambda-2)(\lambda+3) = 0$$

$$\lambda_1 = -2, \quad \lambda_2 = 2, \quad \lambda_3 = -3$$

La solution générale est:

$$\boxed{y(x) = C_1 e^{-2x} + C_2 e^{2x} + C_3 e^{-3x}}$$

✓

Qu. 2. *Résoudre le problème aux valeurs initiales.*
Solve the initial value problem.

$$y'' + y = 2\cos x, \quad y(0) = 1, \quad y'(0) = 0.$$

$Y_h = y'' + y = 0$

posons $y = e^{\lambda x}$

$\hookrightarrow \lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\lambda = \pm i \qquad \alpha = 0 \quad \beta = 1$

$Y(h) = C_1 \cos x + C_2 \sin x$

Y_p est dans Y_h donc il faut multiplier par x

$Y_p = ax\cos x + bx\sin x$

$Y'_p = a\cos x - ax\sin x + b\sin x + bx\cos x$

$Y''_p = -a\sin x - a\sin x - ax\cos x + b\cos x - bx\sin x + b\cos x$

$y''_p + y_p = -2a\sin x - a\cancel{x\cos x} + 2b\cos x - b\cancel{x\sin x} + \cancel{ax\cos x} + \cancel{bx\sin x}$

$= -2a\sin x + 2b\cos x = 2\cos x$

$a = 0$

$2b = 2$

$b = 1$

$Y_h + Y_p = C_1\cos x + C_2\sin x + x\sin x$

$Y_{gen} = C_1\cos x + C_2\sin x + x\sin x$

$Y_g(0) = C_1 + 0 + 0 = 1 \Rightarrow C_1 = 1$

$\boxed{Y_g = \cos x + x\sin x}$

$y'_g(x) = -C_1\sin x + C_2\cos x + \sin x + x\cos x$

$y'_g(x) = 0 + C_2 + 0 + 0 = 0$

$C_2 = 0$

Qu. 2. *Résoudre le problème aux valeurs initiales.*
Solve the initial value problem.

$$y'' + y = 2\cos x, \quad y(0) = 1, \quad y'(0) = 0.$$

$y_h(x)$: $\lambda^2 + 1 = 0$ $\quad \lambda = \pm i$

$\lambda^2 = -1$

$y_h(x) = c_1 \cos x + c_2 \sin x$

$y_p(x)$: variation des paramètres

$y_p = c_1(x) \cos x + c_2(x) \sin x$

$$\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} 0 \\ 2\cos x \end{bmatrix}$$

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2\cos x \end{bmatrix}$$

$$\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2\cos x \end{bmatrix} \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

$c_1' = 2\cos x(-\sin x) \Rightarrow c_1 = \cos^2 x$

$c_2' = 2\cos^2 x \Rightarrow c_2 = x + \frac{1}{2} \sin 2x$

$y_p = \cos^3 x + x \sin x + \frac{1}{2} \sin 2x \sin x$

$y_g = c_1 \cos x + c_2 \sin x + \cos^3 x + x \sin x + \frac{1}{2} \sin 2x \sin x$

$y_g(0) = 1 = c_1 \cos 0 + c_2 \sin 0 + \cos^3 0 + 0 \sin 0 + \frac{1}{2} \sin 0 \sin 0 = c_1 + 1 \Rightarrow c_1 = 0$

$y_g'(x) = -c_1 \sin x + c_2 \cos x + 3\cos^2 x \sin x + \sin x + x \cos x + 2\cos 2x \sin x + \frac{1}{2} \sin 2x \cos x$

$y_g'(0) = 0 = -c_1 \sin 0 + c_2 \cos 0 - 3\cos^2 0 \sin 0 + \sin 0 + 0 \cos 0 + 2\cos 2 0 \sin 0 + \frac{1}{2} \sin 0 \cos 0$

$0 = c_2$

$y_g(x) = \cos^3 x + x \sin x + \frac{1}{2} \sin 2x \sin x$ ✓

Qu. 4. *Trouver la solution générale.*

Find the general solution.

$$\boldsymbol{y}' = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix} \boldsymbol{y}.$$

$$\det(A-\lambda I) = \det\begin{bmatrix} -1-\lambda & 1 \\ 4 & -1-\lambda \end{bmatrix} \Rightarrow (-1-\lambda)^2 - 4 = 0$$

$$(\lambda^2+2\lambda+1)-4=0$$

$$\lambda^2+2\lambda-3=0$$

$$(\lambda-1)(\lambda+3)=0$$

$$\lambda_1=1 \quad \lambda_2=-3$$

$$(A-\lambda_1 I)\vec{u} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \quad \begin{matrix} -2u_1+u_2=0 \\ 2u_1=u_2 \end{matrix}$$

posons $u_1=1$, on trouve $u_2=2$

$$(A-\lambda_2 I)\vec{v} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \begin{matrix} 2u_1+u_2=0 \\ 2u_1=-u_2 \end{matrix}$$

posons $u_1=1$ on trouve $u_2=-2$

on obtient alors la solution générale

$$y(x) = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{x} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-3x}$$

✓

✓

Qu. 5 *Construire le polynôme de Lagrange qui interpole la fonction*
Construct Lagrange's polynomial which interpolates the function

$$f(x) = e^{2x}\cos 3x, \quad \text{en / at} \quad x_0 = 0,\ x_1 = 0.3,\ x_2 = 0.6.$$

$$L_0 = \frac{(x-x_1)}{(x_0-x_1)}\frac{(x-x_2)}{(x_0-x_2)} = \frac{(x-0{,}3)}{-0{,}3}\frac{(x-0{,}6)}{-0{,}6}$$

$$L_1 = \frac{(x-x_0)}{(x_1-x_0)}\frac{(x-x_2)}{(x_1-x_2)} = \frac{x}{0{,}3}\frac{(x-0{,}6)}{-0{,}3}$$

$$L_2 = \frac{(x-x_0)}{(x_2-x_0)}\frac{(x-x_1)}{(x_2-x_1)} = \frac{x}{0{,}6}\frac{(x-0{,}3)}{0{,}3}$$

angle en radian

$$p(x) = f(x_0)L_0 + f(x_1)L_1 + f(x_2)L_2$$

$$= L_0 + 1{,}13264721\,L_1 - 0{,}7543375195\,L_2$$

$$= 5{,}5\overline{5}(x-0{,}3)(x-0{,}6) + -12{,}584969(x(x-0{,}6))$$
$$+ -4{,}190763397(x(x-0{,}3))$$

$$=$$

Qu. 6. *Compléter le tableau de différences divisées :*
Complete the divided difference table:

i	x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	3.2	22.0			
			8.400		
1	2.7	17.8		2.856	
			$\boxed{2.118}$		−0.528
2	1,0	14.2		$\boxed{2.011}$	
			$\boxed{6.342}$		$\boxed{-4.273}$
3	4.8	38.3		$\boxed{-10.382}$	
			$\boxed{-41.413}$		
4	5.6	5.17			

Construire le polynôme de degré 3 qui interpole les données aux 4 points de $x_0 = 3.2$ *à* $x_3 = 4.8$.
Construct the cubic interpolating polynomial which interpolates the data at the 4 points $x_0 = 3.2$ to $x_3 = 4.8$.

$$f(x) = 22.0 + (x-3.2)\cdot 8.400 + \frac{(x-3.2)(x-2.7)}{2}\cdot 2.856 + \frac{(x-3.2)(x-2.7)(x-1.0)}{6}\cdot(-0.528)$$

$$f(x) = 22.0 + 8.4(x-3.2) + 1.428(x-3.2)(x-2.7) - 0.088(x-3.2)(x-2.7)(x-1.0)$$