MAT2784B

2008.03.29

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DEVOIR 7

6.9 $f(t) = e^{-2t}\cosh t = e^{-2t}\left(\frac{e^{t}}{2} + \frac{e^{-t}}{2}\right)$

$$= \frac{1}{2}\left[e^{-t} + e^{-3t}\right]$$

$$\mathcal{L}(f(t))(s) = \frac{1}{2}\left[\mathcal{L}(e^{-t}) + \mathcal{L}(e^{-3t})\right]$$

$$= \frac{1}{2}\int_0^\infty e^{-st}e^{-t}\,dt + \frac{1}{2}\int_0^\infty e^{-st}e^{-3t}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s+1)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(s+3)t}\,dt$$

$$= \frac{1}{2}\left(\frac{-1}{s+1}\right)\left[e^{-(s+1)t}\right|_0^\infty + \frac{1}{2}\left(\frac{-1}{s+3}\right)\left[e^{-(s+3)t}\right|_0^\infty$$

$$= \frac{1}{2}\left(\frac{1}{s+1}\right) + \frac{1}{2}\left(\frac{1}{s+3}\right) = \frac{1}{2}\left[\frac{1}{s+1} + \frac{1}{s+3}\right]$$

$$= \frac{1}{2}\left(\frac{s+3+s+1}{(s+1)(s+3)}\right) = \frac{1}{2}\left(\frac{2s+4}{(s+1)(s+3)}\right) = \frac{s+2}{(s+1)(s+3)}$$

$$\boxed{\mathcal{L}(f(t))(s) = \frac{s+2}{(s+1)(s+3)}}$$

D 7.2

6.15

$$F(s) = \frac{4(s+1)}{s^2-16} = \frac{4s}{s^2-4^2} + \frac{4}{s^2-4^2}$$

$$= 4\left[\frac{s}{s^2-4^2}\right] + 4\left[\frac{1}{s^2-4^2}\right]$$

$$= 4\,\mathcal{L}(\cosh(4t))(s) + \mathcal{L}(\sinh(4t))(s)$$

$$\Rightarrow \boxed{f(t) = \mathcal{L}^{-1}(F(s)) = 4\cosh(4t) + \sinh(4t)}$$

6.29

$$F(s) = \frac{e^{-3s}}{s^2(s-1)} = e^{-3s}\left[\frac{1}{s^2(s-1)}\right]$$

Soit $F(s) = \dfrac{1}{s^2(s-1)}$, alors $\mathcal{L}^{-1}(e^{-3s}F(s)) = y(t-3)u(t-3) = f(t)$

par le théorème 6.7

$$F(s) = \frac{1}{s^2(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}$$

$$1 = A[s(s-1)] + B(s-1) + C(s^2)$$
$$1 = (A+C)s^2 + (B-A)s - B$$

$$\left.\begin{aligned} A+C &= 0 \\ B-A &= 0 \\ -B &= 1 \end{aligned}\right\} \quad \begin{aligned} A &= -1 \\ B &= -1 \\ C &= 1 \end{aligned} \quad \Rightarrow F(s) = \frac{-1}{s} - \frac{1}{s^2} + \frac{1}{s-1}$$

$\Rightarrow y(t) = \mathcal{L}^{-1}(-1/s) + \mathcal{L}^{-1}(-1/s^2) + \mathcal{L}^{-1}(1/s-1)$

$= -1 - t + e^t$

$y(t-3) = -1-(t-3)+e^{t-3}$

$= e^{t-3} - t + 2$

$\Rightarrow f(t) = y(t-3)\,u(t-3)$ et $u(t-3) = \begin{cases} 0 & 0 \le t \le 3 \\ 1 & t > 3 \end{cases}$

$$\boxed{\text{Donc } f(t) = \begin{cases} 0 & 0 \le t \le 3 \\ e^{t-3} - t + 2 & t > 3 \end{cases}}$$

<u>6.42</u> $y'' + y = \sin 3t$ $\quad y(0) = 0$, $y'(0) = 0$

Soit $Y(s) = \mathcal{L}(y)(s)$

$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(\sin 3t)$

$s^2 Y(s) - s(y(0)) - y'(0) + Y(s) = \dfrac{3}{s^2+3^2}$

$(s^2+1)\,Y(s) = \dfrac{3}{s^2+9}$ puisque $y(0) = y'(0) = 0$

$Y(s) = \dfrac{3}{(s^2+9)(s^2+1)} = \dfrac{A}{s^2+9} + \dfrac{B}{s^2+1}$

$$\Rightarrow \quad 3 = A(s^2+1) + B(s^2+9)$$
$$3 = (A+B)s^2 + (A+9B)$$

$$\begin{aligned} A+B &= 0 \\ A+9B &= 3 \end{aligned} \quad \Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 9 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 3/8 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -3/8 \\ 0 & 1 & 3/8 \end{array}\right)$$

$$\Rightarrow A = -3/8\,, \quad B = 3/8$$

$$\Rightarrow Y(s) = \frac{-3}{8}\left(\frac{1}{s^2+9}\right) + \frac{3}{8}\left(\frac{1}{s^2+1}\right)$$

$$y(t) = -\frac{3}{8}\left(\frac{1}{3}\sin 3t\right) + \frac{3}{8}\left(\sin 3t\right)$$

$$\boxed{y(t) = \frac{3}{8}\sin t - \frac{1}{8}\sin 3t}$$

(la solution est tracée à la suite du # 6.45)

**6.45** $\quad y'' + 5y' + 6y = 3e^{-2t} \qquad y(0) = 0, \quad y'(0) = 1$

Soit $Y(s) = \mathcal{L}(y)(s)$

$$\mathcal{L}(y'') + 5\mathcal{L}(y') + 6\mathcal{L}(y) = 3\mathcal{L}(e^{-2t})$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 5\left[sY(s) - y(0)\right] + 6Y(s) = \frac{3}{s+2}$$

$$(s^2+5s+6)Y(s) - (s+5)\,y(0) - y'(0) = \frac{3}{s+2}$$

$$(s^2+5s+6)Y(s) - 1 = \frac{3}{s+2}$$

$$(s+2)(s+3)\,Y(s) = \frac{s+5}{s+2} \quad \Rightarrow \quad Y(s) = \frac{s+5}{(s+2)^2(s+3)}$$

$$\frac{s+5}{(s+2)^2(s+3)} = \frac{A}{(s+2)} + \frac{B}{(s+2)^2} + \frac{C}{(s+3)}$$

$$s+5 = (s+2)(s+3)A + (s+3)B + (s+2)^2 C$$
$$s+5 = (s^2+5s+6)A + (s+3)B + (s^2+4s+4)C$$
$$s+5 = (A+C)s^2 + (5A+B+4C)s + (6A+3B+4C)$$

$$A+C=0$$
$$5A+B+4C=1$$
$$6A+3B+4C=5$$

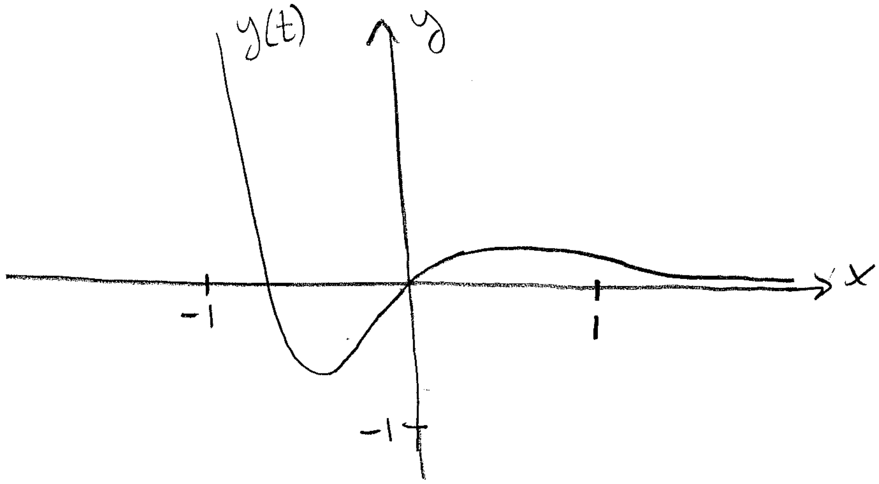
$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 5 & 1 & 4 & 1 \\ 6 & 3 & 4 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & -2 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right) \qquad \begin{aligned} A &= -2 \\ B &= 3 \\ C &= 2 \end{aligned}$$

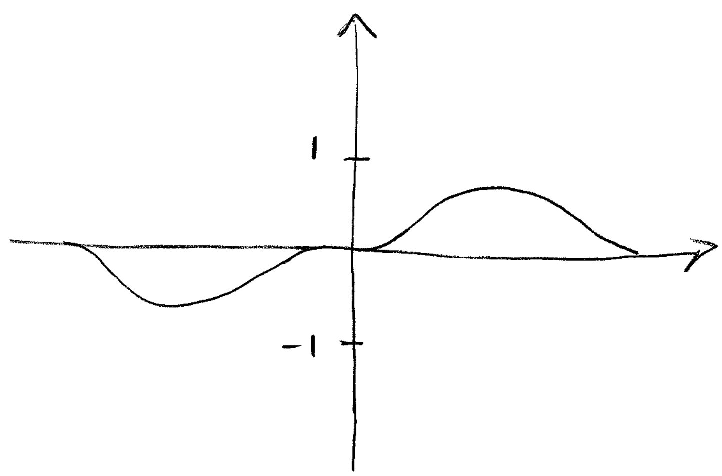
$$\Rightarrow \quad Y(s) = \frac{-2}{s+2} + \frac{3}{(s+2)^2} + \frac{2}{s+3}$$

$$Y(s) = -2\left(\frac{1}{s-(-2)}\right) + 3\left(\frac{1}{[s-(-2)]^2}\right) + 2\left(\frac{1}{s-(-3)}\right)$$

$$\boxed{y(t) = -2e^{-2t} + 3te^{-2t} + 2e^{-3t}}$$




graphe du # 6.42 :



<u>6.47</u> $y'' - 4y' + 4y = t^3 e^{2t}$  $y(0)=0$, $y'(0)=0$

Sait $\mathcal{L}(y)(s) = Y(s)$

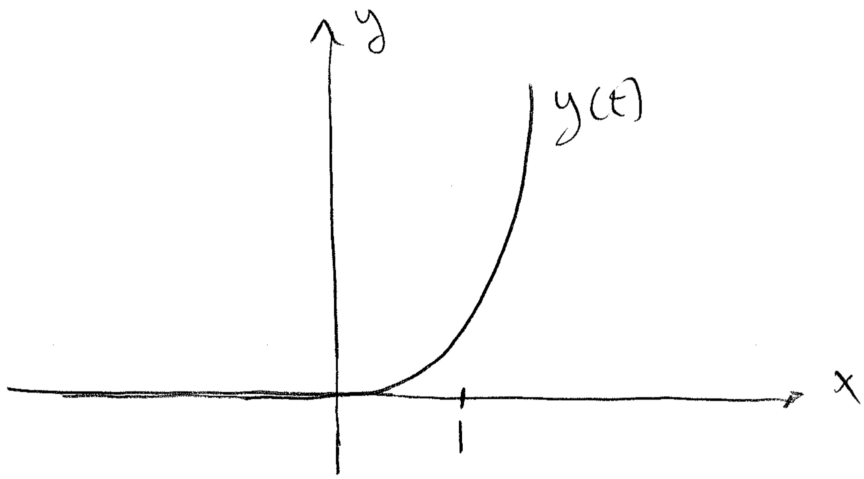
$$\mathcal{L}(y'') - 4\mathcal{L}(y') + 4\mathcal{L}(y) = \mathcal{L}(t^3 e^{2t})$$

$$s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0} - 4\left[sY(s) - \cancel{y(0)}^{0}\right] + 4Y(s) = \frac{3!}{(s-2)^4}$$

$$(s^2 - 4s + 4)\,Y(s) = \frac{6}{(s-2)^4} \Rightarrow Y(s) = \frac{6}{(s-2)^6}$$

$$y(t) = 6\left[\frac{1}{(6-1)!}\, t^{6-1} e^{2t}\right]$$

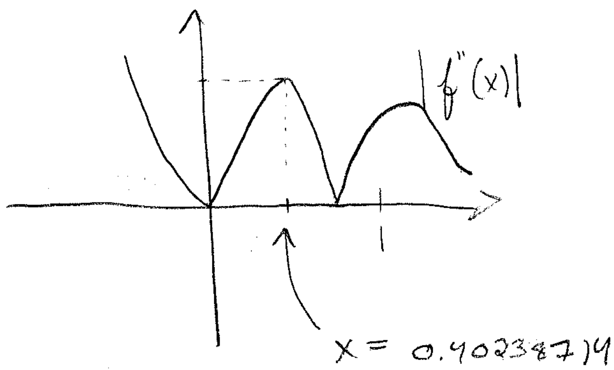
$$\boxed{y(t) = \frac{1}{20}\, t^5 e^{2t}}$$



## 10.7

$$f(x) = \frac{1}{1+x^3} \qquad f''(x) = \frac{12x^4 - 6x}{(1+x^3)^3}$$

On a que $M = \max_{0 \le x \le 1} |f''(x)|$




donc $M = \max_{0 \le x \le 1} |f''(x)| = f''(0.40238714) = 1.7375069$

$$\left| \frac{(b-a)h^2}{12} f''(\varepsilon) \right| < \frac{h^2}{12} M = h^2 (0.1447922417) \le 10^{-4}$$

$$\Rightarrow \underline{h = 0.0262801006}$$

donc $\frac{1}{h} = 38.05157 \le \underline{n = 39}$

donc $\int_0^1 \frac{dx}{1+x^3} = \frac{h}{2}\left[ 0 + 2f(h) + \dots + 2f(38h) + f(39h) \right]$

$= \boxed{0.8356}$

10.11

$$I = \int_1^{1.5} x^2 \ln x \, dx$$

$$h = 1.5 - 1 = 0.5$$

$$\Rightarrow h_1 = 0.5, \quad h_2 = \frac{h}{2} = 0.25, \quad h_3 = \frac{h}{4} = 0.125$$

$R_{1,1}$ est obtenue par la méthode des trapèzes avec un pas de $h_1 = 0.5$

$$R_{1,1} = \frac{h_1}{2}\left[f(1) + f(1.5)\right] = 0.25\left[0 + (1.5)^2 \ln(1.5)\right]$$

$$= 0.2280741233$$

$R_{2,1}$ obtenue avec pas $h_2 = 0.25$

$$R_{2,1} = \frac{h_2}{2}\left[f(1) + 2f(1.25) + f(1.5)\right] = 0.2012025114$$

$R_{3,1}$ obtenue avec pas $h_3 = 0.125$

$$R_{3,1} = \frac{h_3}{2}\left[f(1) + 2f(1.125) + 2f(1.25) + 2f(1.375) + f(1.5)\right]$$

$$= 0.1944944732$$

$$R_{2,2} = R_{2,1} + \frac{R_{2,1} - R_{1,1}}{3} = 0.1922453074$$

$$R_{3,2} = R_{3,1} + \frac{R_{3,1} - R_{2,1}}{3} = 0.1922584604$$

$$R_{3,3} = R_{3,2} + \frac{R_{3,2} - R_{2,2}}{15} = 0.1922593373$$

$$\Rightarrow I \approx R_{3,3} = \boxed{0.1922593373}$$

Table d'intégration de Romberg

| | | |
|---|---|---|
| 0.2280741230 | | |
| 0.2012025114 | 0.1922453074 | |
| 0.1944944732 | 0.1922584604 | 0.1922593373 |