NOM/NAME SOLUTIONS

Nº D'ÉT/ST.#

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2784 B

Test mi-session 2
Durée: 90 min
Place: MPT 103
19 mars 2008
17:30–19:00

Midterm 2
Time: 90 min
Place: MPT 103
19th of March 2008
17:30–19:00

Prof.: Rémi Vaillancourt

Instructions:

(a) *À livre fermé. Tout type de calculatrices autorisé.*
Closed book. All types of calculators are allowed.
(b) *Répondre sur le questionnaire. Réponses numériques dans les boîtes.*
Answer on the question sheets. Fill-in boxes with numerical answers.
(c) *Les 7 questions sont d'égale valeur.*
All 7 questions have the same value.
(d) *Donner le détail de vos calculs.*
Show all computation.
(e) *Une feuille couleur de tables sera distribuée.*
A one-page table on colored paper will be distributed.
(f) *Tous les angles sont en RADIANS. Tester et ajuster votre calculatrice.*
All angles are in RADIAN measures. Test and adjust your calculators.

$$\sin 1.123456789 = 0.90160112364453$$

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
TOTAL	/60

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Qu. 1. *Résoudre le problème aux valeurs initiales.*
Solve the initial value problem.

$$y' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} y, \qquad y(0) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = 1 + \lambda^2 - 2\lambda - 4$$
$$= \lambda^2 - 2\lambda - 3$$
$$= (\lambda+1)(\lambda-3)$$
$$\lambda_1 = -1 \qquad \lambda_2 = 3$$

1er vecteur propre. $\lambda_1 = -1$

$$(A - (-1)I)u = 0 \Rightarrow \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2u_1 + u_2 = 0 \qquad u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
$$u_1 = \frac{-u_2}{2} \quad \text{on pose } u_1 = 1$$

2e vecteur propre $\lambda_2 = 3$

$$(A - 3I)v = 0 \Rightarrow \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2v_1 + v_2 = 0 \Rightarrow v_1 = \frac{v_2}{2} \qquad \text{on pose } v_1 = 1$$

$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

solution générale : $y(x) = C_1 e^{-x}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{3x}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$y(0) = \begin{bmatrix} C_1 \\ -2C_1 \end{bmatrix} + \begin{bmatrix} C_2 \\ 2C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \Rightarrow \begin{array}{l} C_1 + C_2 = 2 \\ -2C_1 + 2C_2 = -3 \end{array}$$

$$C_1 = 2 - C_2 \Rightarrow -4 + 2C_2 + 2C_2 = -3 \Rightarrow 4C_2 = 1$$
$$C_2 = \frac{1}{4}$$
$$C_1 = 2 - \frac{1}{4} = \frac{7}{4}$$

Solution particulière : $\boxed{y(x) = \frac{7}{4} e^{-x}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \frac{1}{4} e^{3x}\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

Qu. 2. *Trouver la solution générale séries de puissance de centre $a = 0$.*

Find the general solution by power series with centre $a = 0$.

$$y'' + x^2y' + xy = 0.$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + \cdots$$

$$x^2 y' = a_1 x^2 + 2a_2 x^3 + 3a_3 x^4 + 4a_4 x^5 + 5a_5 x^6 + 6a_6 x^7$$

$$xy = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7$$

$$0 = 2a_2 + x(6a_3 + a_0) + x^2(12a_4 + a_1 + a_1) + x^3(20a_5 + 2a_2 + a_2)$$
$$+ x^4(30a_6 + 3a_3 + a_3) + \cdots$$

x^0 : $2a_2 = 0 \Rightarrow a_2 = 0$.

x^1 : $6a_3 + a_0 = 0 \Rightarrow a_3 = -a_0/6$. avec a_0 et a_1 indeterminé.

x^2 : $12a_4 + 2a_1 = 0 \Rightarrow a_4 = -a_1/6$.

x^3 : $20a_5 + 3a_2 = 0 \Rightarrow a_5 = -3a_2/20$

x^4 : $30a_6 + 4a_3 = 0 \Rightarrow a_6 = -4a_3/30 = -\frac{4}{30} \times \left(\frac{a_0}{6}\right) = -\frac{2}{90} a_0$

$\boxed{x^s : \quad a_{s+2}(s+1)(s+2) + s\, a_{s-1}} \Rightarrow \boxed{a_{s+2} = \frac{-s\, a_{s-1}}{(s+1)(s+2)}}$

$$y(x) = a_0 + a_1 x - \frac{a_0}{6} x^3 - \frac{a_1}{6} x^4 - \frac{3a_2}{20} x^5 - \frac{4a_3}{30} x^6 \cdots$$

Qu. 3. *Trouver les 4 premiers termes de la série de Fourier–Legendre.*

Find the first 4 terms of the Fourier–Legendre expansion.

$$\cos x = \sum_{m=0}^{\infty} a_m P_m(x), \qquad -1 \le x \le 1.$$

Les 4 premiers polynômes de LEGENDRE / The first 4 Legendre polynomials:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), \quad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right).$$

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx.$$

$$a_0 = \frac{1}{2} \int_{-1}^{1} \underbrace{\cos x}_{\text{paire}}\, dx = \int_0^1 \cos x\, dx = \sin x \Big|_0^1 \qquad (*)$$

$$= \sin 1 = 0,841470985$$

$$a_1 = \frac{3}{2} \int_{-1}^{1} \underbrace{\underbrace{P_1(x)}_{\text{impair}} \underbrace{\cos x}_{\text{pair}}}_{\text{impair}}\, dx = 0$$

$$a_2 = \frac{5}{2} \cdot \frac{1}{2} \int_{-1}^{1} \underbrace{(3x^2 - 1)\cos x}_{\text{pair}}\, dx = \frac{5}{2} \int_0^1 (3x^2 - 1)\cos x\, dx$$

$$= \frac{5}{2} \cdot 3 \int_0^1 x^2 \cos x\, dx - \frac{5}{2} \int_0^1 \cos x\, dx \qquad \text{par } (*)$$

$$= \frac{15}{2} \int_0^1 x^2 \cos x\, dx - \frac{5}{2} \sin 1$$

$$I = \int_0^1 x^2 \cos x\, dx = x^2 \sin x \Big|_0^1 - 2 \int_0^1 x \sin x\, dx$$

$$= \sin 1 + 2\left[x \cos x \Big|_0^1 - \int_0^1 \cos x\, dx \right] \qquad \text{par } (*)$$

$$= \sin 1 + 2 \cos 1 - 2 \sin 1$$

$$a_2 = \frac{15}{2}\left[2 \cos 1 - \sin 1 \right] - \frac{5}{2} \sin 1$$

$$= 15 \cos 1 - 10 \sin 1 = 0,856842.$$

$$a_3 = \frac{7}{2} \int_{-1}^{1} \underbrace{\underbrace{P_3(x)}_{\text{impair}} \underbrace{\cos x}_{\text{pair}}}_{\text{impair}}\, dx = 0$$

Qu. 4(a) *Évaluer l'intégrale par la quadrature gaussienne à 3 points.*

Evaluate the integral by Gauss' three-point quadrature.

$$I = \int_{0.3}^{1.9} \sin x^2 \, dx.$$

Attention : *Les angles en radians.* / Angles in radians.

$$\int_a^b \longrightarrow \int_{-1}^{1} \text{ avec / with } x = \frac{b-a}{2} t + \frac{b+a}{2}.$$

Quadrature gaussienne / Gaussian quadrature :

$$\int_{-1}^{1} f(x)\,dx \sim \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right).$$

$$I = \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{b-a}{2} t + \frac{b+a}{2} \right) dt$$

$$I = 0.8 \int_{-1}^{1} f(0.8t + 1.1)\, dt$$

$$I = 0.8 \int_{-1}^{1} \sin(0.8t + 1.1)^2 \, dt$$

$$I = 0.8 \left[\frac{5}{9} \sin\left(0.8\left(-\sqrt{\tfrac{3}{5}}\right) + 1.1\right)^2 + \frac{8}{9} \sin\left(0.8(0) + 1.1\right)^2 + \frac{5}{9} \sin\left(0.8\left(\sqrt{\tfrac{3}{5}}\right) + 1.1\right)^2 \right]$$

$$I = 0.8 \left[0,127038 + 0,831658668 + 0,10181614 \right]$$

$$\underline{I = 0,848406686}$$

Qu. 4(b) *Trouver le rayon de convergence de la série.*

Find the radius of convergence of the series.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{k^n} x^{2n}$$

$$a_n = \frac{(-1)^n}{k^n} \; ;$$

$$\frac{1}{R} = \lim\sup |a_n|^{1/2n}$$

$$\frac{1}{R} = \lim \left| \frac{(-1)^n}{k^n} \right|^{1/2n} = \lim \left| \frac{(-1)^{1/2}}{k^{1/2}} \right|$$

$$\frac{1}{R} = \frac{1}{|k|^{1/2}} \Rightarrow \boxed{R = |k|^{1/2}} \quad \text{pour } x < |k|^{1/2}$$

Qu. 5(a). *Trouver les transformées de Laplace de $f(t)$.*
Find the Laplace transforms of $f(t)$.

$$f(t) = 3\cosh 3t + 4\sin 2t.$$

$$F(s) = 3\cdot\left(\frac{s}{s^2-9}\right) + 4\left(\frac{2}{s^2+4}\right)$$

$$f(t) = e^{-3t}\left(t^2 + \cos 5t\right).$$

$$\frac{2}{(s+3)^3} + \frac{(s+3)}{(s+3)^2+25}$$

Qu. 5(b). *Trouver les transformées de Laplace inverses de $F(s)$.*
Find the inverse Laplace transforms of $F(s)$.

$$F(s) = \frac{5(s+4)}{s^2-16}. \quad = \frac{5s}{s^2-16} + \frac{20}{s^2-16}$$

$$\Rightarrow f(t) = 5\cosh 4t + \frac{20}{4}\left(\sinh 4t\right)$$

$$\boxed{f(t) = 5\cosh 4t + 5\sinh 4t}$$

$$F(s) = \frac{1+e^{-2s}}{s+3}. \quad = \frac{1}{(s+3)} + e^{-2s}\cdot\frac{1}{(s+3)}$$

$$\underline{f(t) = e^{-3t} + u(t-2)\cdot e^{-3(t-2)}.}$$

Qu. 6. *Résoudre par transformation de Laplace.*

Solve by Laplace transforms.

$$y'' + y = \delta(t-2), \quad y(0) = 3, \quad y'(0) = 5.$$

$$s^2 Y(s) - sY(0) - Y'(0) + Y(s) = e^{-2s}$$

$$(s^2+1)\,Y(s) = e^{-2s} + 3s + 5$$

$$Y(s) = \frac{e^{-2s}}{(s^2+1)} + \frac{3s}{(s^2+1)} + \frac{5}{(s^2+1)}$$

$$\underline{y(t) = u(t-2)\sin(t-2) + 3\cos t + 5\sin t}$$

✓

Qu. 7. *Soit le terme de l'erreur de la méthode des trapèzes pour* $\int_a^b f(x)\,dx$:

The truncation error of the composite trapezoidal rule for $\int_a^b f(x)\,dx$ is:

$$-\frac{(b-a)h^2}{12}f''(\xi).$$

Déterminer les valeurs de h et n pour approcher l'intégrale suivante à 10^{-4} près par la méthode des trapèzes :

Determine the values of h and n to approximate the following integral to 10^{-4} by the composite trapezoidal rule :

$$\int_2^5 \ln(2x)\,dx.$$

$\ln(ab) = \ln a + \ln b$
$\ln 2x = \ln 2 + \ln x$

$f(x) = \ln(2x) = \ln 2 + \ln x$
$f'(x) = 1/x$
$f''(x) = -1/x^2$

$$M = \max_{2 \leq x \leq 5} |f''(\xi)| = |f''(2)| = 0.25$$

$$\frac{(b-a)h^2}{12} M = \frac{3h^2}{12} \cdot 0.25 \leq 10^{-4}$$

$$h^2(0.0625) \leq 10^{-4}$$

$$h^2 \leq 16 \times 10^{-4}$$

$$h \leq 4 \times 10^{-2}$$

or $h = \frac{b-a}{n} = \frac{3}{n} \Rightarrow n \geq \frac{3}{h} = 75$

Donc $\boxed{n = 75.}$

$h = \frac{3}{75} \Rightarrow \boxed{h = 0.04}$