NOM/NAME SOLUTIONS

Nº d'Ét/St #

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

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Test mi-session 1
Durée: 90 min
Place: MPT 103
13 février 2008
17:30–19:00

MAT 2784 B

Midterm 1
Time: 90 min
Place: MPT 103
13th of February 2008
17:30–19:00

Prof.: Rémi Vaillancourt

Instructions:

(a) *À livre fermé. Tout type de calculatrices autorisé.*
Closed book. All types of calculators are allowed.

(b) *Répondre sur le questionnaire. Réponses numériques dans les boîtes.*
Answer on the question sheets. Fill-in boxes with numerical answers.

(c) *Les 6 questions sont d'égale valeur.*
All 6 questions have the same value.

(d) *Donner le détail de vos calculs.*
Show all computation.

(e) *Une feuille couleur de tables sera distribuée.*
A one-page table on colored paper will be distributed.

(f) *Tous les angles sont en RADIANS. Tester et ajuster votre calculatrice.*
All angles are in RADIAN measures. Test and adjust your calculators.

$$\sin 1.123456789 = 0.90160112364453$$

1.	/10
2.	/10
3.	/10
4.	/10
5.	/10
6.	/10
TOTAL	/60

L'équation différentielle homogène du 1er ordre,
The first order homogeneous differential equation,

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

admet un facteur d'intégration
admits an integrating factor

$$\mu(x) \quad \text{ou/or} \quad \mu(y)$$

selon que
according to

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x) \Longrightarrow \mu(x) = e^{\int f(x)\,dx},$$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y) \Longrightarrow \mu(y) = e^{-\int g(y)\,dy}.$$

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Qu. 1. *Trouver un facteur d'intégration, rendre l'équation différentielle exacte et résoudre le problème à valeur initiale.*

Find an integration factor, make the differential equation exact and solve the initial value problem.

$$(x+y^2)\,dx - 2xy\,dy = 0, \qquad y(1) = 3.$$

$M = x + y^2 \quad ; \quad N = -2xy$

$M_y = 2y \; ; \; N_x = -2y \Rightarrow \frac{M_y - N_x}{N} = \frac{4\cancel{y}}{-2x\cancel{y}} = -\frac{2}{x}$

$\mu(x) = e^{\int f(x)\,dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}.$

$x^{-2}(x+y^2)\,dx - (2xy)x^{-2}\,dy = 0$

$\left(x^{-1} + \frac{y^2}{x^2}\right)dx - \frac{2y}{x}\,dy = 0$

$\mu(x,y) = -\int \frac{2y}{x}\,dy + T(x)$

$= -\cancel{2}\frac{y^2}{\cancel{2}x} + T(x) = -\frac{y^2}{x} + T(x)$

$\mu_x = +\cancel{\frac{y^2}{x^2}} + T'(x) = x^{-1} + \cancel{\frac{y^2}{x^2}}$

$T'(x) = \frac{1}{x} \Rightarrow T(x) = \int \frac{1}{x}\,dx \Rightarrow T(x) = \ln x$

$\mu(x,y) = \boxed{-\frac{y^2}{x} + \ln x = c}$ solution générale

Condition initiale: $y = 3 \; ; \; x = 1 \Rightarrow -\frac{9}{1} + \cancel{\ln 1} = c$

$\Rightarrow \boxed{c = -9}$

solution unique : $\boxed{-\frac{y^2}{x} + \ln x = -9}$

Qu. 2. *Résoudre le problème à valeur initiale.*

Solve the initial value problem.

$$y' + \frac{2}{x} y = 12, \qquad y(1) = 2.$$

$$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln x^2} = x^2$$

$$[x^2 y]' = 12x^2$$

$$x^2 y = \int 12x^2 \, dx + c$$

$$x^2 y = \frac{12x^3}{3} + c$$

$$\boxed{x^2 y = 4x^3 + c}$$ solution generale.

Condition initiale: $y = 2$; $x = 1$

$$1^2(2) = 4(1)^3 + c \Rightarrow c = 2 - 4 \Rightarrow \boxed{c = -2}$$

$$x^2 y = 4x^3 - 2$$

$$x^2 y = 2(2x^3 - 1)$$

$$\boxed{y = \frac{2(2x^3 - 1)}{x^2}}$$ solution unique. ✓

Qu. 3. *Résoudre le problème aux valeurs initiales.*

Solve the initial value problem.

$$x^2y'' - xy' + y = 0, \qquad x > 0, \quad y(1) = 1, \quad y'(1) = 3.$$

Posons $y = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1)x^{m-2}$

$x^2\, m(m-1)x^{m-2} - (x\, m x^{m-1}) + x^m = 0$

$x^m \cdot m(m-1) - x^m \cdot m + x^m = 0$

$x^m \left(m(m-1) - m + 1 \right) = 0$

$m^2 - m - m + 1 = 0$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m_1 = m_2 = m = 1$

$\boxed{y(x) = C_1 X + C_2(\ln x)x}$ solution generale.

Condition initiale : $y(1) = 1 \Rightarrow 1 = C_1 + C_2(0) \Rightarrow \boxed{C_1 = 1}$

$y'(x) = C_1 + C_2\left(\ln x + x\left(\frac{1}{x}\right)\right)$

$3 = 1 + C_2(\cancel{\ln 1} + 1)$

$3 = 1 + C_2 \Rightarrow \boxed{C_2 = 2}$

$y(x) = x + 2(\ln x)\, x$

$\boxed{y(x) = x(1 + 2\ln x)}$ solution unique.

✓

Qu. 4. *Trouver la solution générale.*

Find the general solution.

$$y'' + y = \frac{1}{\sin x}.$$

$y_g = y_h + y_p$

équation homogène : $y'' + y = 0$ on pose $y = e^{\lambda x}$

$\lambda^2 + 1 = 0$

$\lambda = \pm i \Rightarrow \lambda_1 = 0 + (1)i$

$\lambda_2 = 0 - (1)i$

$\boxed{y_h = C_1 \cos x + C_2 \sin x}$

Variations des paramètres : $y_p = C_1(x)\cos x + C_2(x)\sin x$

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sin x} \end{bmatrix}$$

① $C_1'\cos x + C_2'\sin x = 0 \Rightarrow C_1' = -\frac{C_2'\sin x}{\cos x} = -C_2'\tan x$

② $-C_1'\sin x + C_2'\cos x = \frac{1}{\sin x}$

$C_2'\frac{\sin^2 x}{\cos x} + C_2'\cos x = \frac{1}{\sin x} \Rightarrow C_2'\left(\frac{1}{\cos x}\right) = \frac{1}{\sin x}$

$C_2' = \frac{\cos x}{\sin x} \Rightarrow C_2 = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x|$

$C_1' = -C_2'\tan x = -\frac{\cos x}{\sin x} \times \frac{\sin x}{\cos x} = -1$

$C_1 = -x$

$$\boxed{y_g = C_1\cos x + C_2\sin x - x\cos x + \ln(|\sin x|)\sin x}$$

Qu. 5. *Itérer 5 fois la récurrence de point fixe à 6 décimales près. Trouver l'ordre de convergence. Les angles sont en radians.*

Iterate 5 times the fixed point recurrence. Use at leas 6 decimals. Find the order of convergence of the method. Angles are in radians.

$$x_{n+1} = g(x_n), \quad x_0 = 1, \quad \text{with} \quad g(x) = \cos x.$$

$g(x) = \cos x$.

$g'(x) = -\sin x$.

$$x_{n+1} = \cos x_n = g(x_n)$$

n	x_n	Ecart $\lvert x_{n+1} - x_n \rvert$
0	1.000000	0.459697695
1	0.540302305	0.31725091
2	0.857553215	0.20326342 5
3	0.65428979	0.13919056 8
4	0.793480358	0.09211158 5
5	0.701368773	

$|g'(p)| = -\sin(0.7) = |0.6442| \neq 0$

$0 < |g'(p)| < 1 \Rightarrow$

L'ordre de convergence est 1.

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Qu. 6. *Compléter le tableau pour la récurrence de point fixe :*

Fill the boxes for the fixed point iteration:

$$x_{n+1} = \sqrt{2x_n + 3} \equiv g(x_n).$$

n	x_n	Δx_n	$\Delta^2 x_n$
1	$x_1 = \boxed{2.0000}$		
		$\boxed{0.64575}$	
2	$x_2 = \boxed{2.64575}$		$\boxed{-0.41201}$
		$\boxed{0.23374}$	
3	$x_3 = \boxed{2.87949}$		

Accélérer la convergence par la méthode d'Aitken.

Accelerate the convergence by Aitken's process.

$$a_1 = x_1 - \frac{(\Delta x_1)^2}{\Delta^2 x_1} = \boxed{3.0120945}$$

$$a_1 = 2 - \frac{(0.64575)^2}{(-0.41201)}$$

$$2 - (-1.0120945)$$

Accélérer la convergence par la méthode de Steffensen

Accelerate the convergence by Steffensen's process.

$$s_1 = a_1,$$

$$z_1 = g(s_1) = \boxed{3.004028801}$$

$$z_2 = g(z_1) = \boxed{3.001342633}$$

$$s_2 = s_1 - \frac{(z_1 - s_1)^2}{z_2 - 2z_1 + s_1} = \boxed{3.000001346}$$

$$s_2 = 3.0120945 - \frac{(3.004028801 - 3.0120945)^2}{3.001342633 - 2(3.004028801) + 3.0120945}$$

$$s_2 = s_1 - \frac{(6.5055500036 \times 10^{-5})}{5.379531 \times 10^{-3}}$$

$$s_2 = s_1 - 0.012093154$$

$$s_2 = 3.000001346$$