D8 1

SOLUTIONS

MAT 2784 A
Devoir 8

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$\boxed{6.48}$ $y'' + 4y = \begin{cases} 1 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$ $\quad y(0) = 0 \quad y'(0) = -1$

on pose : $\mathcal{L}(y) = Y(s)$ et $\mathcal{L}(g) = G(s)$

$$\mathcal{L}(y'' + 4y) = s^2 Y(s) - sY(0) - y'(0) + 4Y(s)$$
$$= (s^2 + 4)\, Y(s) + 1 = G(s)$$

$$\Rightarrow Y(s) = \frac{G(s) - 1}{s^2 + 4}$$

$$g(t) = 1 - u(t-1)1 + u(t-1)0 = 1 - u(t-1)1$$
$$G(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

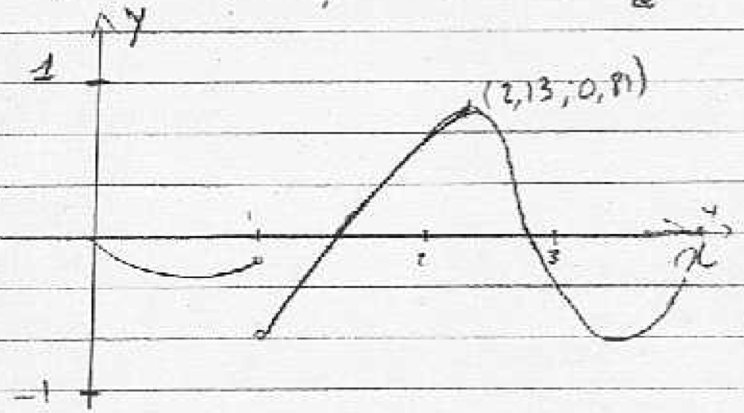
$$\Rightarrow Y(s) = \frac{1}{s(s^2+4)} - \frac{e^{-s}}{s(s^2+4)} - \frac{1}{s^2+4}$$

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4} = \frac{As^2 + 4A + Bs^2 + Cs}{s(s^2+4)}$$

$$\Rightarrow C = 0 \quad A = 1/4 \quad B = -1/4$$

donc $Y(s) = \frac{1}{4s} - \frac{s}{4(s^2+4)} - e^{-s}\left[\frac{1}{4s} - \frac{s}{4(s^2+4)}\right] - \frac{1}{s^2+4}$

$$y(t) = \begin{cases} \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{2}\sin 2t & \text{si } 0 \le t \le 1 \\ -1/4 \cos 2t + \frac{1}{4}\cos(2t-2) - \frac{1}{2}\sin 2t & \text{si } t > 1 \end{cases}$$



[6,51] $y'' + 4y' = u(t-1)$ $\quad y(0)=0$ $\quad y'(0)=0$

On pose $\mathcal{L}(y) = Y(s)$ $\quad \mathcal{L}(u) = G(s)$

$$\mathcal{L}(y''+4y') = s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0)$$
$$= s^2Y(s) + 4sY(s)$$
$$= (s^2+4s)Y(s) = G(s)$$

$$\Rightarrow Y(s) = \frac{G(s)}{s^2+4s}$$

$$g(t) = u(t-1) \Rightarrow G(s) = \frac{e^{-s}}{s}$$

Donc $Y(s) = \dfrac{e^{-s}}{s(s^2+4s)} = \dfrac{e^{-s}}{s^2(s+4)}$

$$\frac{1}{s^2(s+4)} = \frac{As+B}{s^2} + \frac{C}{s+4} = \frac{As^2+Bs+4As+4B + Cs^2}{s^2(s+4)}$$

$\Rightarrow 4B = 1 \Rightarrow B = 1/4$

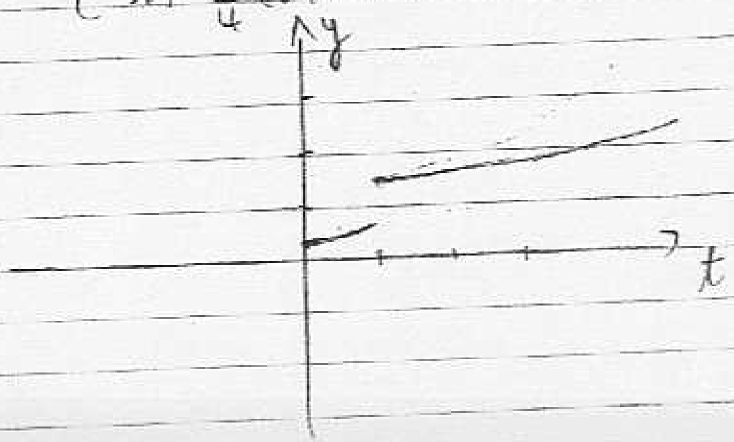
$A + C = 0 \Rightarrow C = 1/16$

$B + 4A = \quad A = -B/4 = -1/16$

$$\Rightarrow Y(s) = e^{-s}\left(\frac{-1/16\, s + 1/4}{s^2} + \frac{1}{16(s+4)}\right) = e^{-s}\left(\frac{-1(s+4)}{16s^2} + \frac{1}{16(s+4)}\right)$$

$$Y(s) = -\frac{e^{-s}}{16s} + \frac{4e^{-s}}{16s^2} + \frac{e^{-s}}{16(s+4)}$$

$$\Rightarrow y(t) = \begin{cases} \frac{1}{4}(t+1) + e^{-4(t+1)} & 0 \le t < 1 \\ 1 + \frac{1}{4}(t+1) + e^{-4(t+1)} & t \ge 1 \end{cases}$$



$\boxed{6.53}$ $y'' - y = \sin t + \delta(t - \pi/2)$ $\quad y(0) = 3.5 \quad y'(0) = -3,5$

$\mathcal{L}(y'' - y) = s^2 Y(s) - s\,y(0) - y'(0) - Y(s) = (s^2-1)\,Y(s) - 3,5s + 3,5 = G(s)$

$(s^2-1)\,Y(s) = 3,5s - 3,5 + G(s)$

$g(t) = \sin t + \delta(t - \frac{\pi}{2})$

alors $G(s) = \frac{1}{s^2+1} + e^{-\pi/2 s}$

donc $(s^2-1)\,Y(s) = 3,5s - 3,5 + \frac{1}{s^2+1} + e^{-\frac{\pi}{2}s}$

$\Rightarrow Y(s) = \frac{1}{(s^2-1)}\left[3,5s - 3,5 + \frac{1}{s^2+1} + e^{-\pi/2\, s}\right] = \frac{3,5s}{s^2-1} - \frac{3,5}{s^2-1} + \frac{1}{(s^2-1)(s^2+1)} + \frac{e^{-\pi/2}}{s^2-1}$

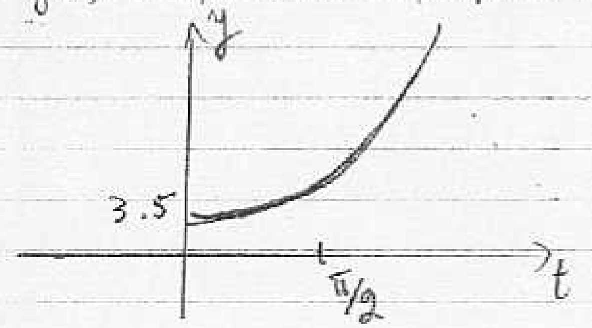
or: $\frac{1}{(s^2-1)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{s+1}$

$= (A+C+D)s^3 + (B-C+D)s^2 + (-A+C+D)s + (-B-C+D)$

$\Rightarrow A = 0, \quad B = -\frac{1}{2}, \quad C = -\frac{1}{4}, \quad D = \frac{1}{4}$

$Y(s) = \frac{3,5s}{s^2-1} - \frac{3,5}{s^2-1} - \frac{1}{2(s^2+1)} - \frac{1}{4(s+1)} + \frac{1}{4(s-1)} + \frac{e^{-\pi/2 s}}{s^2-1}$

$\Longrightarrow y(t) = 3,5\cosh t - 3,5\sinh t - \frac{1}{2}\sin t - \frac{1}{4}e^{-t} + \frac{1}{4}e^{t} + \sinh(t - \frac{\pi}{2})$ $\cdot u(t - \pi/2)$



D 8,4

6.58 $y(t) = t + e^t + \int_0^t y(\tau)\cosh(t-\tau)\, d\tau$

$y(t) = t + e^t + y(t)\cosh(t)$

$\Rightarrow Y(s) = \frac{1}{s^2} + \frac{1}{s-1} + Y(s)\left(\frac{s}{s^2-1}\right)$

$Y(s) - Y(s)\left(\frac{s}{s^2-1}\right) = \frac{1}{s^2} + \frac{1}{s-1}$

$Y(s) = \frac{s^2-1}{s^2(s^2-s-1)} + \frac{(s+1)}{(s^2-s-1)}$

Fraction partielle :

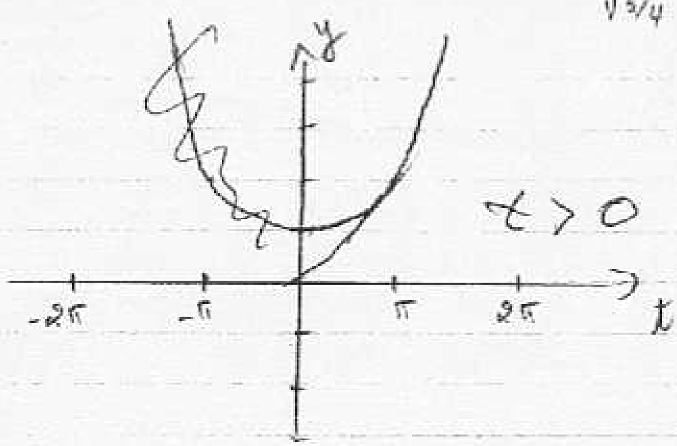
$\frac{s^2-1}{s^2(s^2-s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2-s-1} = (A+C)s^3 + (-A+B+D)s^2 + (-A-B)s - B$

$\Rightarrow A = -1 \quad , \quad B = 1 \quad , \quad C = 1 \quad , \quad D = -1$

donc $Y(s) = \frac{-1}{s} + \frac{1}{s^2} + \frac{s-1}{s^2-s-1} + \frac{s+1}{s^2-s-1} = -\frac{1}{s} + \frac{1}{s^2} + \frac{2s}{s^2-s-1}$

$Y(s) = \frac{-1}{s} + \frac{1}{s^2} + 2\,\frac{s-1/2}{(s-\frac{1}{2})^2 - 5/4} + \frac{1}{\sqrt{5/4}} \cdot \frac{\sqrt{5/4}}{(s-1/2)^2 - 5/4}$

$\Rightarrow y(t) = -1 + t + 2e^{-t/2}\cosh(\sqrt{5/4}\,t) + \frac{1}{\sqrt{5/4}}\sinh(\sqrt{\frac{5}{4}}\,t)$

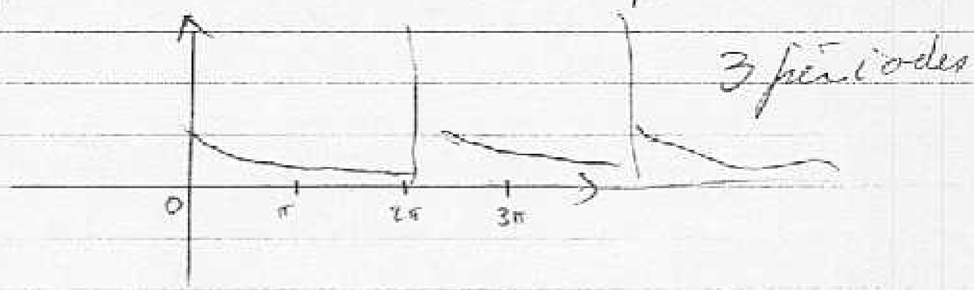


$y(0) = 0$!!

$t > 0$

D 8.5

$\boxed{6.62}$ $f(t) = e^{-t}$ $\quad 0 < t < 2\pi$ $\quad$ période 2π



$f(t) = e^{-t}$

$$\mathcal{L}(f)(s) = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} e^{-t}\, dt$$

$$F(s) = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-t(s+1)}\, dt = \frac{1}{1-e^{-2\pi s}} \left[-\frac{1}{s+1} e^{-t(s+1)} \right]_0^{2\pi}$$

$$\boxed{F(s)} = -\frac{e^{-2\pi(s+1)}}{(1-e^{-2\pi s})(s+1)} + \frac{1}{(1-e^{-2\pi s})(s+1)} = \boxed{\frac{1-0{,}0018\, e^{-2\pi s}}{(s+1)(1-e^{-2\pi s})}}$$

$\boxed{11.10}$

$$\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

$$\begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ 0 & g_{22} & g_{23} \\ 0 & 0 & g_{33} \end{bmatrix} = \begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix}$$

$$\left.\begin{array}{l} g_{11}^2 = 4 \Rightarrow g_{11} = 2 \\ g_{21} g_{11} = 10 \Rightarrow g_{21} = 5 \\ g_{11} g_{31} = 8 \Rightarrow g_{31} = 4 \\ g_{21}^2 + g_{22}^2 = 26 \Rightarrow g_{21} = 1 \\ g_{21} g_{31} + g_{22} g_{32} = 26 \Rightarrow g_{32} = 6 \\ g_{31}^2 + g_{32}^2 + g_{33}^2 = 61 \Rightarrow g_{33} = 3 \end{array}\right\} \quad G = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix}$$

$A = GG^T$

$\det A = \det G \det G^T = \det G^2 = 36$

$$Gy = b \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

$$y_1 = 22 \quad y_2 = 18 \quad y_3 = 6 \Rightarrow y = \begin{bmatrix} 22 \\ 18 \\ 6 \end{bmatrix}$$

$$G^T x = y = \begin{bmatrix} 2 & 5 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 18 \\ 6 \end{bmatrix} \Rightarrow \left.\begin{array}{l} x_1 = -8 \\ x_2 = 6 \\ x_3 = 2 \end{array}\right\} \boxed{x = \begin{bmatrix} -8 \\ 6 \\ 2 \end{bmatrix}}$$

$\boxed{11.12}$ $-2x_1 + x_2 + 5x_3 = 22$ $\qquad x_1^{(0)} = 1$

$x_1 + 4x_2 + 2x_3 = 13$ $\qquad$ avec $\quad x_2^{(0)} = 1$

$7x_1 + 2x_2 - x_3 = -6$ $\qquad x_3^{(0)} = 1$

On arrange le système pour avoir une diagonale dominante :

$7x_1 + 2x_2 - x_3 = -6 \qquad x_1^{(0)} = 1$

$x_1 + 4x_2 + 2x_3 = 13 \qquad x_2^{(0)} = 1$

$-2x_1 + x_2 + 6x_3 = 22 \qquad x_3^{(0)} = 1$

$x_1^{(n+1)} = 1/7\left(-6 - 2x_2^{(n)} + x_3^{(n)}\right) \qquad x_1^{(0)} = 1$

$x_2^{(n+1)} = 1/4\left(13 - x_1^{(n+1)} - 2x_3^{(n)}\right) \qquad x_2^{(0)} = 1$

$x_3^{(n+1)} = 1/6\left(22 - 2x_1^{(n+1)} - x_2^{(n+1)}\right) \qquad x_3^{(0)} = 1$

$n=0 \Rightarrow \quad x_1^{(1)} = 1/7(-6-2+1) = -1$

$x_1^{(1)} = 1/4(13+1-2) = 3$

$x_3^{(1)} = 1/6(22-2-3) = 2,8333$

$n=1 \Rightarrow \quad x_1^{(2)} = 1/7\left(-6 - 2x_2^{(1)} + x_3^{(1)}\right) = -1,30952381$

$x_2^{(2)} = 1/4\left(13 - x_1^{(2)} - 2x_3^{(1)}\right) = 2,160714286$

$x_3^{(2)} = 1/6\left(22 - 2x_1^{(2)} - x_2^{(2)}\right) = 2,870039683$

$n=2 \Rightarrow \quad x_1^{(3)} = -1,064484127$

$x_2^{(3)} = 2,08110119$

$x_3^{(3)} = 2,964988426$

Donc : $\lim_{n \to \infty} x^{(n)} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$