Devoir #5
MAT 2784 B
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#4.2 $\vec{y}' = A\vec{y}$ où $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

On a $\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Trouvons les valeurs propres de A

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} + 4\begin{vmatrix} 0 & 2-\lambda \\ -1 & 0 \end{vmatrix}$$

$$= (2-\lambda)\left[(2-\lambda)^2 + 4(2-\lambda)\right] = 0$$

$$= (2-\lambda)\left((2-\lambda)^2 + 4\right) = 0$$

$$= (2-\lambda)(4-4\lambda+\lambda^2+4) = 0$$

$$= (2-\lambda)(\lambda^2-4\lambda+8) = 0$$

$\boxed{\lambda_1 = 2}$

$$\frac{-(-4) \pm \sqrt{(-4)^2-4(1)(8)}}{2(1)}$$

$\boxed{\lambda_{2,3} = 2 \pm 2i}$

Maintenant, trouvons les vecteurs propres

$\underline{\lambda_1 = 2}$

$$(A-2I)\vec{u} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \vec{0}$$

Où $4u_3 = 0$ $\quad u_3 = 0$
$-u_1 = 0$ $\quad u_1 = 0$

Un vecteur propre n'égale pas le vecteur nul, alors posons $u_2 = 1$

[redacted]

Nous avons le vecteur propre $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\underline{\lambda = 2-2i}$

$$(A-(2-2i))\vec{v} = \begin{bmatrix} 2i & 0 & 4 \\ 0 & 2i & 0 \\ -1 & 0 & 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \vec{0}$$

① $2iv_1 + 4v_3 = 0$

② $2iv_2 = 0 \Rightarrow v_2 = 0$

③ $-v_1 + 2iv_3 = 0 \Rightarrow v_1 = 2iv_3$

① et ③ se répète $(\frac{1}{2}i)$ de différence ①→③

Posons $v_3 = \frac{1}{2} \Rightarrow v_1 = i$ et $v_2 = 0$

Alors, on a un deuxième vecteur propre $\vec{v} = \begin{bmatrix} i \\ 0 \\ 1/2 \end{bmatrix}$

$\underline{ou}$ $\vec{v} = \begin{bmatrix} 2i \\ 0 \\ 1 \end{bmatrix}$ mieux écrit (pas de fraction)

$\underline{\lambda = 2+2i}$

$$(A-(2+2i))\vec{w} = \begin{bmatrix} -2i & 0 & 4 \\ 0 & -2i & 0 \\ -1 & 0 & -2i \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \vec{0}$$

① $-2iw_1 + 4w_3 = 0$

② $-2iw_2 = 0 \Rightarrow w_2 = 0$

③ $-w_1 - 2iw_3 = 0 \Rightarrow w_1 = 2iw_3$

① et 3 se répète $(-\frac{1}{2}i)$ de différence ①→③

Posons $w_3 = -\frac{1}{2} \Rightarrow w_1 = i$ et $w_2 = 0$

Alors, on a un troisième vecteur propre $\vec{w} = \begin{bmatrix} i \\ 0 \\ -1/2 \end{bmatrix}$

ou $\vec{w} = \begin{bmatrix} 2i \\ 0 \\ -1 \end{bmatrix}$ mieux écrit (pas de fraction)

Donc, on a ~~les~~ solutions indépendantes suivantes:

Seulement $\vec{u}$ et $\vec{w}$ seront utile pour exprimer la solution ou $\vec{w}$ et $\vec{v}$ prenons $\vec{u}$ et $\vec{w}$

$$y_1 = e^{2x}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$y_2 = e^{(2+2i)x}\begin{bmatrix} 2i \\ 0 \\ +1 \end{bmatrix} = e^{2x} e^{2ix}\begin{bmatrix} 2i \\ 0 \\ -1 \end{bmatrix}$$

$$= e^{2x}\begin{bmatrix} 2i \\ 0 \\ -1 \end{bmatrix}(\cos 2x + i\sin 2x)$$

réelle: $e^{2x}\begin{bmatrix} -2\sin 2x \\ 0 \\ -\cos 2x \end{bmatrix}$

imaginaire: $e^{2x}\begin{bmatrix} 2\cos 2x \\ 0 \\ -\sin 2x \end{bmatrix}$

$$\boxed{\vec{y}(x) = c_1 e^{2x}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2x}\begin{bmatrix} -2\sin 2x \\ 0 \\ -\cos 2x \end{bmatrix} + c_3 e^{2x}\begin{bmatrix} 2\cos 2x \\ 0 \\ -\sin 2x \end{bmatrix}}$$

#4,3 au verso →

#4.3 $\vec{y}' = A\vec{y}$ où $A = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}$

On a $\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Trouvons les valeurs propres de A

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ 4 & -1-\lambda \end{vmatrix} = (-1-\lambda)(-1-\lambda) - 4 = 0$$

$= \lambda^2 + 2\lambda + 1 - 4 = 0$

$= \lambda^2 + 2\lambda - 3 = 0$ Somme $= 2$

$= (\lambda+3)(\lambda-1) = 0$ produit $= -3$

Donc $\lambda_1 = -3$, $\lambda_2 = 1$

Maintenant trouvons les vecteurs propres

$\underline{\lambda_1 = 3}$ $(A-(3I))\vec{u} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \vec{0}$

① $2u_1 + u_2 = 0$ $\Rightarrow 2u_1 = -u_2$

② $4u_1 + 2u_2 = 0$

① est une équation équivalente à ② $2 \times ① = ②$

Posons $u_2 = 2$ $\Rightarrow u_1 = -1$

On a donc le vecteur propre

$$\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$\lambda_2 = 1$ $\quad (A - 1I)\vec{v} = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \vec{0}$

$$\begin{aligned} &① \; -2u_1 + u_2 = 0 \quad \Rightarrow 2u_1 = u_2 \\ &② \; 4u_1 - 2u_2 = 0 \end{aligned}$$

① est une équation équivalente à ② $\quad -2① = ②$

Posons $\quad u_2 = 2 \Rightarrow u_1 = 1$

On a donc le vecteur propre

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

On a donc les deux solutions indépendantes suivantes

$$y_1 = e^{-3x} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad y_2 = e^{x} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

La solution générale est donc

$$\boxed{\vec{y}(x) = C_1 e^{-3x} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + C_2 e^{x} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

#4.7 au verso →

4.7 $\quad A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \quad f(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ D 5.6

$y' = Ay + f(x)$

$\det(A - \lambda I) = (1-\lambda)(1-\lambda) - 3 = (1 - \lambda + \lambda^2 - \lambda) - 3 = \lambda^2 - 2\lambda - 2 = 0$

$\lambda_1 = 1 + \sqrt{3}$

$\Rightarrow \begin{bmatrix} -\sqrt{3} & 1 \\ 3 & -\sqrt{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$① \times \sqrt{3} + ② \Rightarrow \begin{bmatrix} 0 & 0 \\ 3 & -\sqrt{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3u_1 - \sqrt{3}u_2 = 0$

on pose $u_1 = 1 \Rightarrow 3 - \sqrt{3}u_2 = 0$

$u_2 = \frac{3}{\sqrt{3}} = \sqrt{3}$

$\lambda_2 = 1 - \sqrt{3}$

$\Rightarrow \begin{bmatrix} \sqrt{3} & 1 \\ 3 & \sqrt{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$① \sqrt{3} - ② \begin{bmatrix} 0 & 0 \\ 3 & \sqrt{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3v_1 + \sqrt{3}v_2 = 0$

on pose $v_1 = 1 \Rightarrow 3 + \sqrt{3}v_2 = 0$

$v_2 = \frac{-3}{\sqrt{3}} = -\sqrt{3}$

$u = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} e^{(1+\sqrt{3})x}$

$v = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} e^{(1-\sqrt{3})x}$

$y_h = C_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} e^{(1+\sqrt{3})x} + C_2 \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} e^{(1-\sqrt{3})x}$

$y_p = \begin{bmatrix} a \\ b \end{bmatrix} \quad y_p' = 0$

$y_p = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \Rightarrow \begin{array}{l} a + b = -1 \Rightarrow a = -1 - b \\ 3a + b = -1 \Rightarrow -3 - 3b + b = -1 \end{array}$

$-2b = 2$

$b = -1$

$a = -1 + 1 = 0$

donc $y_p = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$y = y_h + y_p$

$$\boxed{y = C_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} e^{(1+\sqrt{3})x} + C_2 \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} e^{(1-\sqrt{3})x} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}}$$

$\boxed{4.11}$ $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$ $y_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $y' = Ay$ $y(0) = y_0$ D5: 7

$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda) + 3 = \lambda^2 - 6\lambda + 8$

$\lambda_1 = 2$

$\Rightarrow \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3u_1 - u_2 = 0$

on pose $u_1 = 1$

$\Rightarrow u_2 = 3$

$\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda_2 = 4$

$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_1 - v_2 = 0$

on pose $v_1 = 1$

$\Rightarrow v_2 = 1$

$\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

alors $y(x) = C_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4x}$ (solution générale)

$y(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 \\ 3C_1 \end{bmatrix} + \begin{bmatrix} C_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$C_1 = -3/2$ et $C_2 = \frac{7}{2}$

Solution particulière : $\boxed{y(x) = -\frac{3}{2} \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2x} + \frac{7}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4x}}$

5.1) $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} . x^n = f(x)$

$a_n = \frac{(-1)^n}{(2n+1)}$; $a_{n+1} = \frac{(-1)^{n+1}}{2(n+1)+1}$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{2n+3} \cdot \frac{2n+1}{(-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(2n+1)}{2n+3} \right| = \lim_{n\to\infty} \frac{2n+1}{2n+3}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(2+1/n)}{\cancel{n}(2+3/n)} = \frac{2}{2} = 1$$

$\frac{1}{R} = 1 \Rightarrow \boxed{R = 1}$ pour $|x| < 1$

$f'(x) = \frac{(-1)^n . n . x^{n-1}}{2n+1} = \frac{(-1)^n . n . x^n}{x(2n+1)} \Rightarrow a_n = \frac{(-1)^n . n}{x(2n+1)}$

$$\frac{1}{R'} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}(n+1)}{\cancel{x}(2(n+1)+1)} \times \frac{\cancel{x}(2n+1)}{\cancel{(-1)^n} \times n} \right|$$

$$\frac{1}{R'} = \lim_{n\to\infty} \left| \frac{2n^2+3n+1}{2n^2+3n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{n^2}(2+3/n+1/n^2)}{\cancel{n^2}(2+3/n)} \right|$$

$\frac{1}{R'} = \left| \frac{2}{2} \right| \Rightarrow \frac{1}{R'} = 1 \Rightarrow \boxed{R' = 1}$ pour $|x| < 1$

5.7) - $\sum_{n=0}^{\infty} \frac{(-1)^n}{k^n} x^{3n}$

$$a_n = \frac{(-1)^n}{k^n}$$

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n} = \limsup_{n \to \infty} \left| \frac{(-1)^n}{k^n} \right|^{1/3n}$$

$$\frac{1}{R} = \frac{1}{|k|^{1/3}} \Rightarrow \boxed{R = |k|^{1/3}} \quad \text{pour} \quad x < |k|^{1/3}.$$

$$f'(x) = \frac{(-1)^n . 3n}{k^n} . x^{3n-1}$$

Donc $a_n = \frac{(-1)^n . 3n}{k^n}$

$$\frac{1}{R'} = \lim_{n \to \infty} \left| \frac{(-1)^n . 3n}{k^n} \right|^{1/(3n-1)}$$

$$= \lim_{n \to \infty} \left| (-1)^n . 3n \right|^{1/(3n-1)} . \lim_{n \to \infty} \left| \frac{1}{k^n} \right|^{1/(3n-1)}$$

$$= \lim_{n \to \infty} \left| \frac{1}{k^n} \right|^{\left(\frac{1}{n}\right) . \left(\frac{n}{3n-1}\right)}$$

$$= \lim_{n \to \infty} \left| \frac{1}{k} \right|^{n/(3n-1)} = \lim_{n \to \infty} \left(\frac{1}{|k|} \right)^{\frac{\cancel{n}}{\cancel{n}}\left(\frac{1}{3-1/n}\right)}$$

$$\frac{1}{R'} = \lim_{n \to \infty} \left(\frac{1}{|k|} \right)^{1/3} \Rightarrow \boxed{R' = |k|^{1/3}} \quad \text{pour} \quad x < |k|^{1/3}$$

D5.10

10.2) $f(x) = xe^{x}$

$$N_1(h) = N(h) = \frac{1}{2h}\left[f(x_0+h) - f(x_0-h)\right]$$

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \frac{N_1(h/2) - N_1(h)}{3}$$

Affiner la valeur de $f'(\underset{x_0}{1.4})$ avec $h = 0.4$; $h/2$; $h/4$

$$N_1(0.4) = N(0.4) = \frac{1}{0.8}\left[f(1.8) - f(1)\right] = \underline{10.213\,855}$$

$$N_1(0.2) = N(0.2) = \frac{1}{0.4}\left[f(1.6) - f(1.2)\right] = \underline{9.851\,779}$$

$$N_1(0.1) = N(0.1) = \frac{1}{0.2}\left[f(1.5) - f(1.3)\right] = \underline{9.762\,239}$$

Ensuite,

$$N_2(0.4) = N_1(0.2) + \frac{N_1(0.2) - N_1(0.4)}{3} = \underline{9.731\,087}$$

$$N_2(0.2) = N_1(0.1) + \frac{N_1(0.1) - N_1(0.2)}{3} = \underline{9.732\,392}$$

Enfin,

$$N_3(0.4) = N_2(0.2) + \frac{N_2(0.2) - N_2(0.4)}{15} = 9.732479$$

$N_1(0.4) = 10.213\,855$		
$N_1(0.2) = 9.851\,779$	$N_2(0.4) = 9.731\,087$	
$N_1(0.1) = 9.762\,239$	$N_2(0.2) = 9.732\,392$	$N_3(0.4) = 9.732\,479$

10.6) $\displaystyle\int_0^1 \frac{dx}{1+2x^2}$ avec $n = 2m = 10$

$$h = \frac{b-a}{2m} = \frac{1}{10} = 0,1.$$

$x_0 = 0$; $x_1 = 0,1$; $x_2 = 0,2$; $x_3 = 0,3$; $x_4 = 0,4$; $x_5 = 0,5$; $x_6 = 0,6$
$x_7 = 0,7$; $x_8 = 0,8$; $x_9 = 0,9$, $x_{10} = 1$.

$$\int_0^1 \frac{dx}{1+2x^2} = \frac{0,1}{3}\left[1 + \frac{4}{1.02} + \frac{2}{1.08} + \frac{4}{1.18} + \frac{2}{1.32} + \frac{4}{1.5} + \frac{2}{1.72} + \frac{4}{1.98} + \frac{2}{2.28} + \frac{4}{2.62} + \frac{1}{3}\right]$$

$$\int_0^1 \frac{dx}{1+2x^2} = 0.675510 .$$