D4.1

MAT 2784 B

Devoir 4
MAT 2784

RÉMI VAILLANCOURT

$\boxed{3.18}$ $xy'' + y' = 0$ $\qquad y_1(x) = \ln(x)$

Vérifions la solution $\quad x\left(-\frac{1}{x^2}\right) + \frac{1}{x} = -\frac{1}{x} + \frac{1}{x} = 0$

On pose
$$y_2(x) = u(x)\,y_1(x)$$
$$y_2' = u'y_1 + y_1'u$$
$$y_2'' = u''y_1 + y_1'u' + y_1''u + y_1'u' = u''y_1 + 2u'y_1' + y_1''u$$
$$xy_2'' = x(u''y_1 + 2u'y_1' + y_1''u)$$

$$\underbrace{Ly_2}_{0} = \underbrace{uLy_1}_{0} + y_1u' + 2xy_1'u' + xy_1u''$$
$$0 = (y_1 + 2xy_1')u' + (xy_1)u''$$
$$0 = \left(\ln x + 2x\left(\tfrac{1}{x}\right)\right)u' + (x\ln x)u''$$
$$0 = (\ln x + 2)u' + (x\ln x)u''$$
$$v = u' \quad \text{et} \quad v' = u''$$

$$\Rightarrow (\ln x + 2)v = -(x\ln x)\frac{dv}{dx}$$
$$\int \frac{\ln x + 2}{x\ln x}\,dx = -\int \frac{dv}{v}$$
$$\ln v = -\int \frac{2}{x\ln x}\,dx - \int \frac{dx}{x} = -2\ln|\ln x| - \ln x$$
$$\Rightarrow v = e^{-2\ln|\ln x|} \cdot e^{-\ln x}$$
$$v = (\ln x)^{-2} \cdot x^{-1}$$
$$\int v = \int \frac{1}{\ln x^2\, x}\,dx$$
$$\int v = -\frac{1}{\ln x} = u$$

$$y_2 = \ln x \cdot u$$
$$= \ln x\left(\frac{-1}{\ln x}\right) = -1$$

$\boxed{3.25}$ $y'' - y' = e^x \sin x$

$Ly = y'' - y' = e^x \sin x$

$y = y_p(x) + y_h(x)$ $\qquad y_h(x) \to$ solution homogène

donc $y'' - y' = 0$

posons $y = e^{\lambda x}$ $\quad y' = \lambda e^{\lambda x}$ $\quad y'' = \lambda^2 e^{\lambda x}$

$y'' - y' = \lambda^2 e^{\lambda x} - \lambda e^{\lambda x} = 0$

$\lambda^2 - \lambda = 0$

$\lambda(\lambda - 1) = 0$

$\lambda_1 = 0 \quad \lambda_2 = 1$

$y_h(x) = a e^{0x} + b e^{1x}$ avec a, b constantes

$y_h(x) = a + b e^x$

Variation des paramètres

$y_p(x) = C_1(x) + C_2(x) e^x$

$y'_p(x) = 0 + C_2(x) e^x$

$$\begin{bmatrix} y_{P_1}(x) & y_{P_2}(x) \\ y'_{P_1}(x) & y'_{P_2}(x) \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}$$

$$\begin{bmatrix} 1 & e^x \\ 0 & e^x \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ e^x \sin x \end{bmatrix}$$

$C_1'(x) + C_2'(x) e^x = 0$

$C_2'(x) e^x = e^x \sin x$

donc $C_2'(x) = \sin x$

$C_1'(x) + \sin x \, e^x = 0$

$C_1'(x) = -\sin x \, e^x$

$C_1(x) = \int C_1'(x) = \int -e^x \sin x \, dx = -\frac{e^x \sin x}{2} + \frac{e^x \cos x}{2}$

$C_2(x) = \int C_2'(x) = \int \sin x \, dx = -\cos x$

$y_p(x) = -\frac{e^x}{2} \sin x + \frac{e^x}{2} \cos x - e^x \cos x$

$= -\frac{e^x}{2} \sin x - \frac{e^x}{2} \cos x$

$$\boxed{y = y_h + y_p = a + b e^x - \frac{e^x}{2} \sin x - \frac{e^x}{2} \cos x}$$

3.25 $y'' - y' = e^{x}\sin x$

On sait que

$$y_g(x) = y_h(x) + y_p(x)$$

$r(x) = e^{x}\sin x$
$r'(x) = e^{x}\sin x + e^{x}\cos x$
$r''(x) = 2e^{x}\cos x$
$r'''(x) = 2e^{x}\cos x - e^{x}\sin x$
$= (2-1) r'(x)$
$= 1 r'(x)$
Dimension finie = 3

Trouvons $y_h(x)$

$$y'' - y' = 0$$

Le polynôme caractéristique est

$$\lambda^2 - \lambda = 0$$
$$\lambda(\lambda - 1) = 0 \rightarrow \text{équation caractéristique}$$
$$\lambda_1 = 0 \leftarrow \lambda - 1 = 0$$
$$\lambda_2 = 1$$

Donc $\lambda_1 = 0$ et $\lambda_2 = 1$

Alors, la solution homogène est

$$y_h(x) = C_1 e^{0x} + C_2 e^{1x}$$
$$\boxed{y_h(x) = C_1 + C_2 e^{x}}$$

Trouvons $y_p(x)$

coefficients indéterminés

$e^{\alpha x}\cos\beta x + e^{\alpha x}\sin\beta x$

Le second membre à la forme $f(x) = 0e^{x}\cos x + e^{x}\sin x$

Puisque $1+i$ $(\alpha+\beta i)$ n'est pas une racine de l'équation caractéristique on cherche une solution particulière de la forme

$$y_p = Ae^{x}\cos x + Be^{x}\sin x$$

Alors $y_p' = A[e^{x}\cos x - e^{x}\sin x] + B[e^{x}\sin x + e^{x}\cos x]$

$y_p'' = A[e^{x}\cos - e^{x}\sin x - e^{x}\sin x - e^{x}\cos x] + B[e^{x}\sin x + e^{x}\cos x - e^{x}\sin x + e^{x}\cos x]$

$y_p'' = A[-2e^{x}\sin x] + B[2e^{x}\cos x]$

Remplaçons dans l'ED

$$e^x\sin x = -2Ae^x\sin x + 2Be^x\cos x - Ae^x\cos x + Ae^x\sin x - Be^x\sin x - Be^x\cos x$$

$$0e^x\cos x + e^x\sin x = -Ae^x\sin x + Be^x\cos x - Ae^x\cos x - Be^x\sin x$$

Donc $0 = B - A \quad (e^x\cos x) \quad ①$

et $1 = -A - B \quad (e^x\sin x) \quad ②$

Isolons B dans ① $\quad B = A \quad ①'$

①' dans ②

$$1 = -A - A$$
$$1 = -2A$$
$$\boxed{A = \frac{-1}{2}} \qquad \boxed{B = \frac{-1}{2}}$$

Donc la solution particulière est:

$$\boxed{y_p = \frac{-1}{2}e^x\cos x - \frac{1}{2}e^x\sin x}$$

Alors la solution générale est

$$y_g = y_h + y_p$$

$$\boxed{y_g = C_1 + C_2e^x - \frac{1}{2}e^x\cos x - \frac{1}{2}e^x\sin x}$$

3.28) $y''' + y' = x$; $y(0) = 0$; $y'(0) = 1$; $y''(0) = 0$

Posons $y = e^{\lambda x}$

$\lambda^3 + \lambda = 0$

$\lambda(\lambda^2 + 1) = 0$

$\lambda_1 = 0$

$\lambda_2 = i$; $\alpha = 0$; $\beta = 1$

$\lambda_3 = -i$; $\alpha = 0$, $\beta = 1$

$y_H(x) = c_1 e^0 + c_2 e^0 \cos x + c_3 e^0 \sin x$

$y_h(x) = c_1 + c_2 \cos x + c_3 \sin x$

$y_p(x) = x(ax^2 + bx + c)$ COEFF IND.

$y_p(x) = ax^3 + bx^2 + cx$

$y_p'(x) = 3ax^2 + 2bx + c$

$y_p''(x) = 6ax + 2b$

$y_p'''(x) = 6a$

$y_p''' + y_p' = 6a + (3ax^2 + 2bx + c) = r(x) = x$

$3ax^2 + 2bx + (6a + c) = 0x^2 + 1x + 0$

$a = 0$; $2b = 1 \Rightarrow b = 1/2$; $6a + c = 0 \Rightarrow c = 0$

$y_p(x) = \frac{1}{2} x^2$

$$\boxed{y_g(x) = A + B\cos x + C\sin x + 1/2\, x^2}$$

Hilroy

Condition initiale

* $y(0)=0$; $y=0$; $x=0$

$$y = A + B\cos x + C\sin x + \frac{1}{2}x^2$$

$$0 = A + B(1) + 0 + 0 \Rightarrow A+B=0 \Rightarrow \boxed{A=-B}$$

* $y' = B\sin x + C\cos x + x$

$y'=1$; $x=0$

$$1 = -B(0) + C + 0 \Rightarrow \boxed{C=1}$$

* $y'' = B\cos x - C\sin x + 1$

$y''=0$; $x=0$

$$0 = -B + 1 \Rightarrow \boxed{B=1}$$

$$A = -B \Rightarrow \boxed{A=-1}$$

Solution unique : $\boxed{y(x) = -1 + \cos x + \sin x + \frac{1}{2}x^2}$

$\boxed{3.28}$ $y''' + y' = x \qquad y(0) = 0 \quad y'(0) = 1 \quad y''(0) = 0$

$Ly = y''' + y' = 0$

$y = e^{\lambda x} \Rightarrow \lambda^3 + \lambda = (\lambda^2 + 1)\lambda = 0$

$\lambda_1 = 0 \qquad \lambda_{2,3} = \pm i$

$y_h(x) = A + B\cos x + C\sin x$

α) Coeffi. IND. $\quad y_p(x) = ax + bx^2$

$y_p'(x) = a + 2bx \;, \quad y_p''(x) = 2b \quad y_p'''(x) = 0$

$Ly_p = 0 + 2bx = x \Rightarrow a = 0$ et $b = \frac{1}{2}$

$y_p(x) = A + B\cos x + C\sin x + \frac{1}{2}x^2$

β) VAR. DES PAR. $\quad y_p(x) = C_1(x) + C_2(x)\cos x + C_3(x)\sin x$

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array}\begin{bmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{bmatrix}\begin{bmatrix} C_1' \\ C_2' \\ C_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

$(1)+(3) \Rightarrow C_1' = x \Rightarrow C_1(x) = \frac{1}{2}x^2$

$$\left.\begin{array}{c}(2)\\(3)\end{array}\right] \Rightarrow \begin{bmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{bmatrix}\begin{bmatrix} C_2' \\ C_3' \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix} \qquad A\vec{C'} = \begin{bmatrix} 0 \\ x \end{bmatrix}$$

A orthogonale $\Rightarrow A^{-1} = A^t \Rightarrow$

$$\begin{bmatrix} C_2' \\ C_3' \end{bmatrix} = \begin{bmatrix} -\sin x & -\cos x \\ \cos x & -\sin x \end{bmatrix}\begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} -x & \cos x \\ -x & \sin x \end{bmatrix}$$

$C_2(x) = -\int x\cos x\, dx = -x\sin x + \int \sin x\, dx = -x\sin x - \cos x$

$C_3(x) = -\int x\sin x\, dx = x\cos x - \int \cos x\, dx = x\cos x - \sin x$

Donc $y_p(x) = \frac{1}{2}x^2 + (-x\sin x - \cos x)\cos x + (x\cos x - \sin x)\sin x$

$= \frac{1}{2}x^2 - 1$

Solution générale : $y_g(x) = \tilde{A} + B\cos x + C\sin x + \frac{1}{2}x^2 - 1$

$= A + B\cos x + C\sin x + \frac{1}{2}x^2$

$y'(x) = -B\sin x + C\cos x + x$

$y''(x) = -B\cos x - C\sin x + 1$

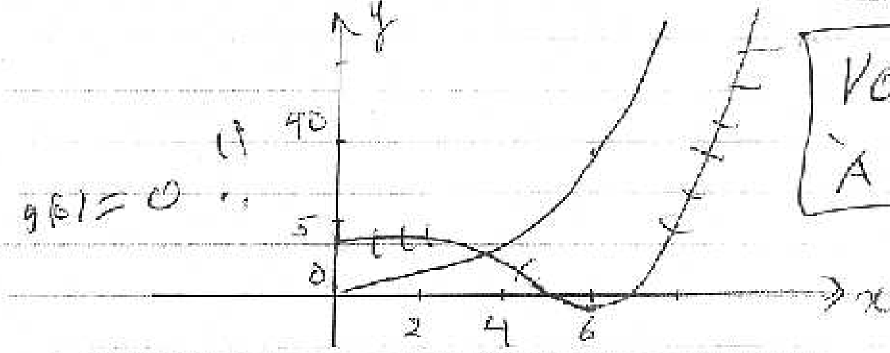
$y(0) = B + A = 0$

$y'(0) = C = 1$

$y''(0) = -B + 1 = 0 \Rightarrow B = 1$

$\Rightarrow A = -1$

Solution unique : $\boxed{y(x) = -1 + \cos x + \sin x + \frac{1}{2}x^2}$



VOIR FIG À LA FIN

$\boxed{3.34}$ $y'' - 2y' + y = \frac{e^x}{x}$

$Ly = y'' - 2y' + y = 0$ on pose $y = e^{\lambda x}$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda_{1,2} = 1$

$y_1(x) = e^x$ $y_2(x) = u(x)\, y_1(x)$

$= x e^x$

$\boxed{y_h(x) = Ae^x + Bxe^x}$

$y_p(x) = C_1(x)e^x + C_2(x)\, x e^x$

$$\begin{bmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{bmatrix}\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^x/x \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} e^x & xe^x \\ 0 & e^x \end{bmatrix}\begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^x/x \end{bmatrix}$$

$e^x C_2' = \frac{e^x}{x} \Rightarrow C_2' = \frac{1}{x} \Rightarrow C_2 = \ln x$

$e^x C_1' + xe^x C_2' = 0 \Rightarrow C_1' = -xC_2' = -1$

$\Rightarrow C_1 = -x$

$\boxed{y_p(x)} = -xe^x + xe^x \ln x = \boxed{(\ln x - 1)\, xe^x}$

Solution générale :

$\boxed{y_g(x) = Ae^x + Bxe^x + (\ln x - 1)\, xe^x}$

$\boxed{3.37}$ $y'' - 2y' + y = \frac{e^x}{x}$ $\quad y(1) = e \quad y'(1) = 0$

Même équation que $\boxed{3.34}$

Solution générale : $y(x) = Ae^x + Bxe^x + (\ln(x) - 1)xe^x$

$$y(x) = Ae^x + \tilde{B}xe^x + xe^x \ln x$$

$$y'(x) = Ae^x + \tilde{B}e^x + \tilde{B}xe^x + e^x \ln x + xe^x \ln x + e^x$$

$$y(1) = Ae + \tilde{B}e = e$$

$$\Rightarrow A + \tilde{B} = 1 \quad \Rightarrow A = 1 - \tilde{B}$$

$$y'(1) = Ae + \tilde{B}e + \tilde{B}e + e = 0$$

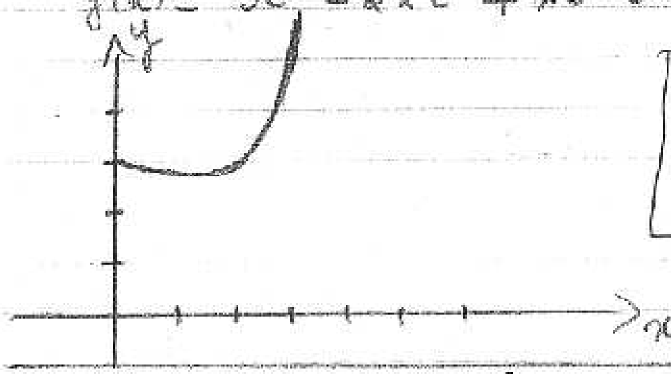
$$A + 2\tilde{B} = -1$$

$$1 - \tilde{B} + 2\tilde{B} = -1 \quad \Rightarrow \tilde{B} = -2$$

$$A = 3$$

Donc Solution unique :

$$y(x) = 3e^x - 2xe^x + xe^x \ln x$$



VOIR FIG. À LA FIN

$\boxed{3.39}$ $Ly = 2x^2y'' + xy' - 3y = 2x^{-3}$ $\quad y(1) = 0 \quad y'(1) = 3$

$$y = y_h + y_p$$

$Ly = 2x^2y'' + xy' - 3y = 0$ $\quad$ posons $y = x^m$

$$\Rightarrow 2m(m-1) + m - 3 = 0$$

$$2m^2 - m - 3 = 0 \quad \Rightarrow m_{1,2} = \frac{1 \pm \sqrt{1+24}}{4}$$

$$m_1 = \frac{3}{2} \qquad m_2 = -1$$

$$y_h(x) = Ax^{3/2} + Bx^{-1}$$

$$y_p(x) = C_1(x)\,x^{3/2} + C_2(x)\,x^{-1}$$
$$y_p'(x) = \frac{3}{2} C_1' x^{1/2} - C_2'(x)\,x^{-2}$$

$$\begin{bmatrix} x^{3/2} & x^{-1} \\ \frac{3}{2}x^{1/2} & -x^{-2} \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ x^{-5} \end{bmatrix}$$

$$x^{3/2} C_1' + C_2'(x^{-1}) = 0 \Rightarrow C_2' = -x^{5/2} C_1'$$
$$\frac{3}{2}x^{1/2}C_1' + x^{-2}C_2' = x^{-5} \Rightarrow C_1' = \frac{2}{5}x^{-11/2}$$

$$\Rightarrow C_2' = -\frac{2}{5}x^{-11/2}\left(x^{5/2}\right) = -\frac{2}{5}x^{-3}$$

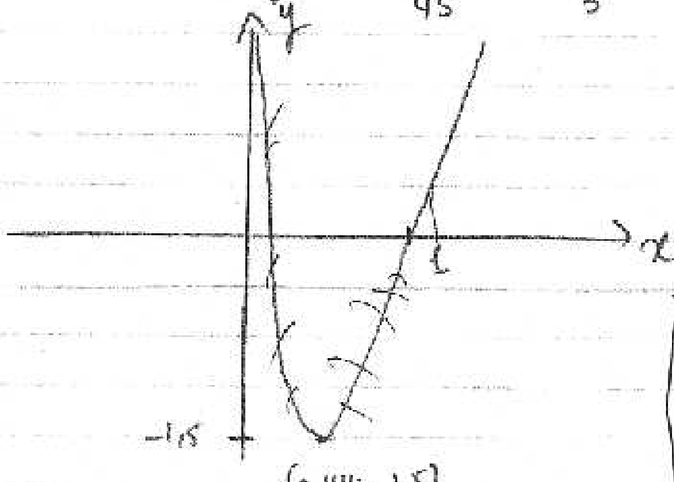
$$C_1(x) = \int \frac{2}{5}x^{-11/2}\,dx = \frac{2}{5}\left(x^{-9/2}\right)\left(-\frac{2}{9}\right) = -\frac{4}{45}x^{-9/2}$$

$$C_2(x) = \int -\frac{2}{5}x^{-3}\,dx = -\frac{2}{5}\left(x^{-2}\right)\left(-\frac{1}{2}\right) = \frac{x^{-2}}{5}$$

$$y_p(x) = \frac{-4}{45}x^{-9/2}\left(x^{3/2}\right) + \frac{x^{-2}}{5}\left(x^{-1}\right) = \frac{1}{9}x^{-3}$$

$$y = y_h + y_p = Ax^{3/2} + Bx^{-1} + \frac{1}{9}x^{-3}$$
$$y'(x) = \frac{3}{2}Ax^{1/2} - Bx^{-2} - \frac{1}{3}x^{-4}$$
$$y(1) = A + B + \frac{1}{9} = 0 \Rightarrow A = -B - \frac{1}{9}$$
$$y'(1) = \frac{3}{2} - B - \frac{1}{3} = 3 \Rightarrow B = -\frac{7}{5} \Rightarrow A = \frac{58}{45}$$

Solution particulière : $y(x) = \frac{58}{45}x^{3/2} - \frac{7}{5}x^{-1} + \frac{1}{9}x^{-3}$



mieux :

$x \geq 1$

VOIR FIG. À LA FIN

9.5

i	x_i	$f(x_i)$	$f(x_i, x_{i+1})$	$f(x_i, x_{i+1}, x_{i+2})$	$f(x_i, x_{i+1}, x_{i+2}, x_{i+3})$
0	3,2	22,0			
			8,400		
1	2,7	17,8		2,856	
			$\boxed{2,118}$		-0,528
2	1,0	14,2		$\boxed{2,012}$	
			$\boxed{6,342}$		$\boxed{-4,273}$
3	4,8	38,3		$\boxed{-10,381}$	
			$\boxed{-41,413}$		
4	5,6	5,17			

X

$$P_3(x) = 22,0 + (x-3,2)(8,400) + (x-3,2)(x-2,7)(2,856) +$$
$$+ (x-3,2)(x-2,7)(x-1,0)(-0,528)$$

9.11

i	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$	$\Delta^4 y_i$
0	0,0	$\boxed{1,0}$				
			$\boxed{0,22140}$			
1	0,2	1,22140		$\boxed{0,04902}$		
			0,27042		$\boxed{0,01086}$	
2	0,4	1,49182		0,05988		$\boxed{0,00238}$
			0,3303		0,01324	
3	0,6	1,82212		0,07312		
			0,40342			
4	0,8	2,22554				

$$n = (x-0)/0,2 = \frac{x}{0,2} = \frac{0,05}{0,2} = 0,25$$

$$P_4(n) = 1 + n(0,22140) + \frac{n(n-1)}{2}(0,04902) + \frac{n(n-1)(n-2)}{6}(0,01086) +$$
$$+ \frac{n(n-1)(n-2)(n-3)}{24}(0,00238)$$

$$P_4(0,25) = 1 + 0,05535 - 0,004596 + 0,0005939 - 0,00008949$$

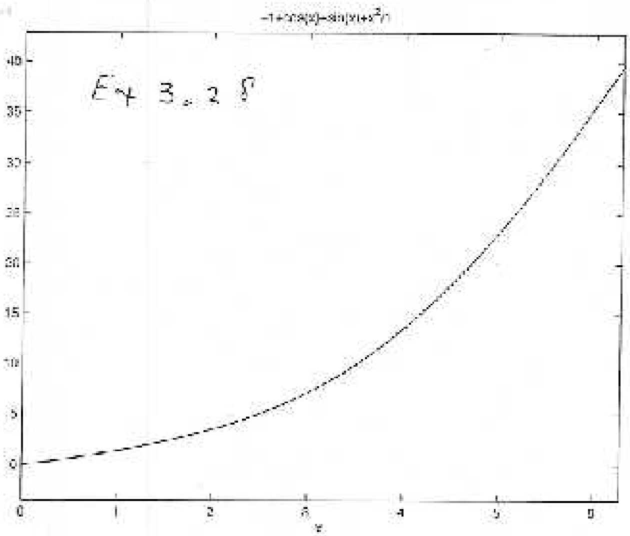
$$\boxed{P_4(0,25) = 1,05126}$$

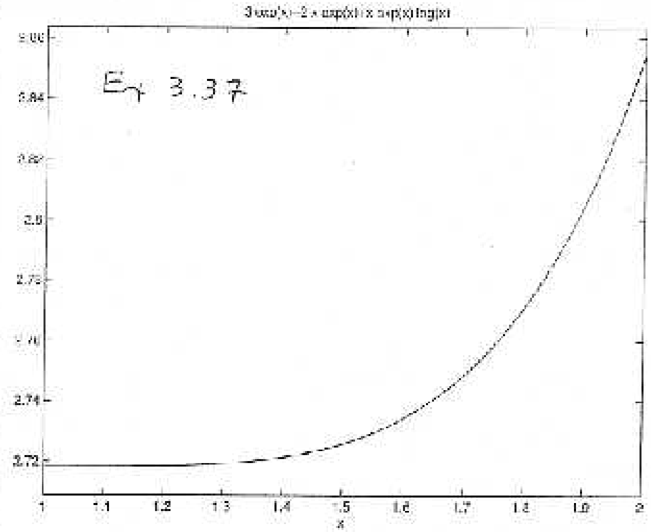
$$P(0,05) = 1 + 0,01107 - 0,0011642 25 + 0,000167 65125 - 0,0000270963$$
$$= 1,010046329$$

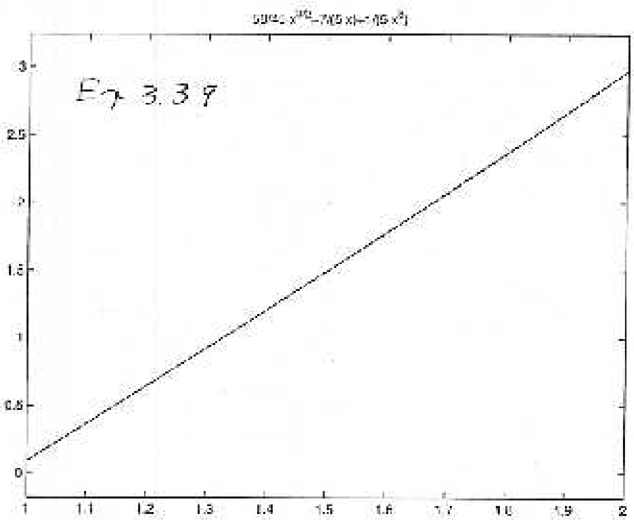
$$\boxed{f(0,05) \simeq 1,010046329}$$

$$f(0,05) \simeq P_4(0,25) = 1,05121$$

4.12







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