MAT 2784B DEV 1

Devoir 1
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Di. 1

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1.2 $y' = \dfrac{xy}{x^2-1} \Rightarrow \dfrac{dy}{dx} = \dfrac{xy}{x^2-1}$

$$\frac{dy}{y} = \frac{x}{x^2-1}\,dx \Rightarrow \int \frac{dy}{y} = \int \frac{x}{x^2-1}\,dx + C$$

$$\Rightarrow \ln|y| = \frac{1}{2}\ln|x^2-1| + C \Rightarrow e^{\ln|y|} = e^{\frac{1}{2}\ln|x^2-1|+C} = e^{\ln|x^2-1|^{1/2}}e^{C}$$

$$\Rightarrow y = k\,(x^2-1)^{1/2}$$

1.8 $x\sin y\,dx + (x^2+1)\cos y\,dy = 0 \qquad y(1) = \frac{\pi}{2}$

Autre sol. : éq. séparable

$M = x\sin y \qquad N = (x^2+1)\cos y$

$M_y = x\cos y \neq N_x = 2x\cos y$ équation pas exacte

On utilise la méthode de facteur d'intégration :

$$f(x) = \frac{M_y - N_x}{N} = \frac{x\cos y - 2x\cos y}{(x^2+1)\cos y} = \frac{-x}{x^2+1}$$

$$\mu(x) = e^{\int \frac{-x}{x^2+1}dx} = e^{-\frac{1}{2}\ln|x^2+1|} = (x^2+1)^{-1/2} = \frac{1}{\sqrt{x^2+1}}$$

Multiplions l'équation différentielle par $\mu(x)$:

$$\mu M\,dx + \mu N\,dy = \frac{x\sin y}{\sqrt{x^2+1}}\,dx + \frac{(x^2+1)\cos y}{\sqrt{x^2+1}}\,dy = 0$$

$$\mu M_y = \frac{x\cos y}{\sqrt{x^2+1}} = \mu N_x = \frac{\cancel{2}x\cos y}{\cancel{2}\sqrt{x^2+1}} \quad \text{équation exacte}$$

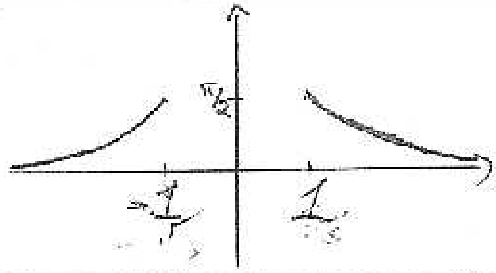
$$u_x = M = \frac{x\sin y}{\sqrt{x^2+1}}$$

$$u(x,y) = \int \frac{x\sin y}{\sqrt{x^2+1}}\,dx + T(y) = \sin y \int \frac{x}{\sqrt{x^2+1}}\,dx + T(y) = \sin y\sqrt{x^2+1} + T(y)$$

$$u_y = \cos y\sqrt{x^2+1} + T'(y) = \frac{(x^2+1)\cos y}{\sqrt{x^2+1}} \Rightarrow T'(y) = 0$$

alors $\left.\begin{array}{l} u(x,y) = \sin y\sqrt{x^2+1} = C \\ \sin\frac{\pi}{2}\sqrt{2} = \sqrt{2} \end{array}\right\} \Rightarrow \sin y = \dfrac{\sqrt{2}}{\sqrt{x^2+1}} \Rightarrow y = \arcsin\sqrt{\dfrac{2}{x^2+1}}$

D1, 2



$\boxed{1.11}$ $xy' = y + \sqrt{y^2 - x^2}$

$$x\frac{dy}{dx} = y + \sqrt{y^2-x^2} \implies \frac{x}{y}\frac{dy}{dx} = 1 + \frac{\sqrt{y^2-x^2}}{y} = 1 + \sqrt{1-\frac{x^2}{y^2}}$$

Substituons $\frac{x}{y} = u \implies x = y \cdot u$

$$dx = y\,du + u\,dy.$$

$$\implies u\,dy = (1+\sqrt{1-u^2})\,dx = (1+\sqrt{1-u^2})[y\,du + u\,dy]$$

$$u\,dy = (1+\sqrt{1-u^2})\,y\,du + (1+\sqrt{1-u^2})\,u\,dy$$

$$u\,dy = [y + y\sqrt{1-u^2}]\,du + u\,dy + (u\sqrt{1-u^2})\,dy$$

$$(y + y\sqrt{1-u^2})\,du = -(u\sqrt{1-u^2})\,dy$$

$$\int (y+y\sqrt{1-u^2})\,du = -\int (u\sqrt{1-u^2})\,dy$$

$$uy + y\left[\frac{1}{2}u\sqrt{1-u^2} + \frac{1}{2}\arcsin u\right] = -uy\sqrt{1-u^2}$$

$$\implies x + y\left[\frac{x}{2y}\sqrt{1-\left(\frac{x}{y}\right)^2} + \frac{1}{2}\arcsin\left(\frac{x}{y}\right)\right] = -x\sqrt{1-\left(\frac{x}{y}\right)^2}$$

#1.14
Autre sol:
eq. à coeff. homogènes de degré 2

$\boxed{1.14}$ $(3x^2 + 9xy + 5y^2)\,dx - (6x^2 + 4xy)\,dy = 0 \qquad y(2) = -6$

$$M_y = 9x + 10y \neq N_x = -12x - 4y$$

$$f(x) = \frac{M_y - N_x}{N} = \frac{9x + 10y + 12x + 4y}{-6x^2 - 4xy} = \frac{21x + 14y}{-6x^2 - 4xy} = \frac{7(3x+2y)}{-2x(3x+2y)} = \frac{7}{-2x}$$

$$\mu(x) = e^{\int 7/_{-2x}\,dx} = e^{\ln x^{-7/2}} = x^{-7/2}$$

$$(3x^{-3/2} + 9x^{-5/2}y + 5x^{-7/2}y^2)\,dx - (6x^{-3/2} - 4x^{-5/2}y)\,dy = 0$$

$$U(x,y) = -\int (6x^{-3/2} + 4x^{-5/2}y)\,dy + T(x) = -(6x^{-3/2}y + 2x^{-5/2}y^2) + T(x)$$

$$U_x = 9x^{-5/2}y + 5x^{-7/2}y^2 + T'(x) = 3x^{-3/2} + 9x^{-5/2}y + 5x^{-7/2}y^2$$

$$\implies T'(x) = 3x^{-3/2} \implies T(x) = -6x^{-1/2}$$

$U(x,y) = -6x^{-1/2}y - 2x^{-1/2}y^2 - 6x^{-1/2}$

Solution générale : $-6x^{-3/2}y - 2x^{-5/2}y^2 - 6x^{-1/2} = C$

or $y(2) = -6 \Rightarrow C = -4,2426$

$\boxed{\text{Donc la solution : } -6x^{-3/2}y - 2x^{-5/2}y^2 - 6x^{-1/2} = -4,2426}$

$\boxed{1.18}$ $(3x^2y^2 - 4xy)y' + 2xy^3 - 2y^2 = 0$

$\Rightarrow (2xy^3 - 2y^2)dx + (3x^2y^2 - 4xy)dy = 0$

$M_y = 6xy^2 - 4y = N_x = 6xy^2 - 4y$ équation exacte

$dU = M dx + N dy$

$U = \int N dy + T(x) = \int (3x^2y^2 - 4xy)dy + T(x)$

$U = x^2y^3 - 2xy^2 + T(x)$

$U_x = 2xy^3 - 2y^2 + T'(x) = 2xy^3 - 2y^2 \Rightarrow T'(x) = 0 \Rightarrow T(x) = 0$

$U(x,y) = x^2y^3 - 2xy^2 = C$

$\boxed{\text{Solution générale : } x^2y^3 - 2xy^2 = C}$

$\boxed{1.22}$ $\frac{2x}{y^3}dx + \frac{y^2 - 3x^2}{y^4}dy = 0$ $y(1) = 1$

$M_y = -6xy^{-4} = N_x = -6xy^{-4}$ équation exacte

$dU = M dx + N dy$

$U = \int M dx + T(y) = \int \frac{2x}{y^3}dx + T(y) = \frac{x^2}{y^3} + T(y)$

$U_y = -3x^2y^{-4} + T'(y) = y^{-2} - 3x^2y^{-4}$

$\Rightarrow T'(y) = y^{-2} \Rightarrow T(y) = -\frac{1}{y}$

$U(x,y) = \frac{x^2}{y^3} - \frac{1}{y}$

Solution générale : $\frac{x^2}{y^3} - \frac{1}{y} = C$ or $y(1) = 1$

$\frac{1^2}{1^3} - \frac{1}{1} = C \Rightarrow C = 0$

$\boxed{\text{Solution unique : } \frac{x^2}{y^3} - \frac{1}{y} = 0}$

5] $x_{n+1} = \sqrt{2x_n+5}$ pour $f(x) = x^2 - 2x - 5 = 0$ converge sur $[2, 4]$

$f(x) = x^2 - 2x - 5 = 0$

$x = \sqrt{2x+5}$

$g(x) = \sqrt{2x_n+5}$

Théorème :

1. la dérivée est croissante sur $[2, 4]$

$g(2) = \sqrt{9} > 2$

$g(4) = \sqrt{13} > 4$ } $g(x)$ continue sur $[2, 4]$ pour $x \in [2, 4]$

2. $g'(x) = \dfrac{1}{\sqrt{(2x_n+5)}}$ existe $\forall x \in [2, 4]$

3. $K = \dfrac{1}{\sqrt{9}}$ (Valeur maximale)

$0 < K < 1 \implies$ converge

6] $f(x) = x^3 - x - 1 = 0$ sur $[1, 2]$ $\quad x_0 = 1$

$f(1) \cdot f(2) = -5$ donc $f(x)$ admet un zéro entre 1 et 2

$x^3 = x + 1$

$x^2 = \dfrac{x+1}{x}$

$x = \sqrt{\dfrac{x+1}{x}} \implies g(x) = \sqrt{\dfrac{x+1}{x}}$

$g'(x) = \dfrac{1}{2}\left(1+\dfrac{1}{x}\right)^{-1/2}(-x^{-2}) = \dfrac{-1}{2x^2} \cdot \dfrac{1}{\sqrt{1+1/x}}$

Toutes les conditions du théorème du point fixe sont satisfaites, même la 3° :

$|g'(x)| = \left| \dfrac{-1}{2x^2} \cdot \dfrac{1}{\sqrt{1+1/x}} \right| \leq K = 0{,}35353391 \quad \forall x \in [1, 2]$

$x_{n+1} = g(x_n) = \left(1 + \dfrac{1}{x}\right)^{1/2}$

Itérations n	racine x_n	écart $\lvert x_n - x_{n+1}\rvert$
0	1	
1	1,414213562	0,414213562
2	1,306562141	0,107650597
3	1,32867109	0,022108125
4	1,323869976	0,004801114
5	1,324900445	0,001030469

La solution à 10^{-2} près est

1,32