NOM / NAME SOLUTIONS

N° d'Ét / St #

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2784 B

Test mi-session 2
Durée: 80 min
Place: MRT 221
21 mars 2007
17:30–18:50

Midterm 2
Time: 80 min
Place: MRT 221
21st of March 2007
17:30–18:50

Prof.: Rémi Vaillancourt

Instructions:

(a) *À livre fermé. Tout type de calculatrices autorisé.*
Closed book. All types of calculators are allowed.

(b) *Répondre sur le questionnaire. Réponses numériques dans les boîtes.*
Answer on the question sheets. Fill-in boxes with numerical answers.

(c) *Les 7 questions sont d'égale valeur. Le test est sur 60 et il y a 10 points de boni.*
All 7 questions have the same value. The test is on 60 and there is a bonus of 10 points.

(d) *Donner le détail de vos calculs.*
Show all computation.

(e) *Une feuille couleur de tables sera distribuée.*
A one-page table on colored paper will be distributed.

(f) *Tous les angles sont en RADIANS. Tester et ajuster votre calculatrice.*
All angles are in RADIAN measures. Test and adjust your calculators.

$$\sin 1.123456789 = 0.90160112364453$$

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
TOTAL	/60

Les 6 premiers polynômes de Legendre :
The first 6 Legendre polynomials:

$$P_0(x) = 1, \qquad P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), \qquad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right),$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right), \quad P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right).$$

Quadratures de GAUSS *à 2 et à 3 points :*
Two- and three-point Gauss quadrature formulae:

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right),$$

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right).$$

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Qu. 1. *Résoudre le système nonhomogène.*
Solve the nonhomogeneous system.

$$y' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} y + \begin{bmatrix} e^x \\ -e^x \end{bmatrix}.$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = -4 + \lambda^2 + 3 = 0$$

$$\lambda^2 - 1 = 0$$

$\lambda = -1$

$$\lambda_{1,2} = \pm 1$$

$$(A + I)\vec{v} = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}$$

$3v_1 - v_2 = 0$

$3v_1 = v_2$

$$\vec{V} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\lambda = 1$

$$(A - I)\vec{w} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \vec{0}$$

$w_1 = w_2 = 1$

$$\vec{W} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Y_h = c_1 e^{-x}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{x}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Y_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x e^x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^x \qquad Y' = Ay + f(x)$$

$$Y'_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}(xe^x + e^x) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^x = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} xe^x + \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} e^x$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} xe^x + \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} e^x = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} xe^x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^x\right) + e^x\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2a_1 - a_2 \\ 3a_1 - 2a_2 \end{bmatrix} xe^x + \begin{bmatrix} 2b_1 - b_2 \\ 3b_1 - 2b_2 \end{bmatrix} e^x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^x$$

Pour xe^x : $a_1 = 2a_1 - a_2$ $\boxed{a_1 = a_2}$

$a_2 = 3a_1 - 2a_2 \Rightarrow 0 = 0$

Pour e^x : $a_1 + b_1 = 2b_1 - b_2 + 1$ $\quad a_1 = b_1 - b_2 + 1$

$a_2 + b_2 = 3b_1 - 2b_2 - 1$ $\quad a_2 = 3b_1 - 3b_2 - 1$

$3b_1 - 3b_2 - 1 = b_1 - b_2 + 1$

$2b_1 - 2b_2 = 2$

$b_1 - b_2 = 1$

Posons $b_1 = 2$

$b_2 = 1$

$a_1 = b_1 - b_2 + 1$

$= 2 - 1 + 1$

$\boxed{a_1 = 2}$

$a_2 = 3b_1 - 3b_2 - 1$

$= 3\cdot 2 - 3\cdot 1 - 1$

$= 6 - 4 = 2$

$a_2 = 2$

$$Y(x) = c_1 e^{-x}\begin{bmatrix}1\\3\end{bmatrix} + c_2 e\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}2\\2\end{bmatrix}xe^x + \begin{bmatrix}2\\1\end{bmatrix}e^x$$

Qu. 2. *Résoudre le problème à valeur initiale.*

Solve the initial value problem.

$$y' = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} y, \quad y(0) = y_0 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = 3 + 4\lambda + \lambda^2 + 2$$
$$= \lambda^2 + 4\lambda + 5$$

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{16 - 4\cdot 5}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = \boxed{-2 \pm i}$$

$\lambda_1 = (-2+i)$

$$(A - (i-2))\vec{v} = \vec{0} \quad \begin{bmatrix} -3-i+2 & 2 \\ -1 & -1-i+2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}$$

$$\underset{L_2 \to L_2 + \frac{L_1}{-1-i}}{\sim} \begin{bmatrix} -1-i & 2 \\ 0 & 1-i+\frac{2}{-1-i} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \qquad \frac{(1-i)(-1-i)+2}{-1-i} = \frac{-1-1+}{-1-i}$$

$$(-1-i)v_1 + 2v_2 = 0$$
$$(-1-i)v_1 = -2v_2 \to (1+i)v_1 = 2V_2$$
$$v_1 = \frac{2V_2}{1+i}$$

$$v_1 = 2 \qquad \vec{v} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$$
$$V_2 = 1+i$$

$$Y = e^{(-2+i)x} \begin{bmatrix} 2 \\ 1+i \end{bmatrix} = e^{-2x} e^{ix} \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$$

$$Y = e^{-2x} \begin{bmatrix} 2 \\ 1+i \end{bmatrix} (\cos x + i \sin x)$$

Réel: $e^{-2x} \begin{bmatrix} 2\cos x \\ \cos x - \sin x \end{bmatrix}$

Imaginaire: $e^{-2x} \begin{bmatrix} 2\sin x \\ \cos x + \sin x \end{bmatrix}$

$$Y_{(x)} = c_1 e^{-2x} \begin{bmatrix} 2\cos x \\ \cos x - \sin x \end{bmatrix} + c_2 e^{-2x} \begin{bmatrix} 2\sin x \\ \cos x + \sin x \end{bmatrix}$$

<u>CI:</u> $Y_{(0)} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\quad\quad \underline{\underline{x=0}}$

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 2\cos 0 \\ \cos 0 - \sin 0 \end{bmatrix} + c_2 \begin{bmatrix} 2\sin 0 \\ \cos 0 + \sin 0 \end{bmatrix}$$

$\sin 0 = 0$
$\cos 0 = 1$

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$1 = 2c_1 + \cancel{0c_2}\ 0 \qquad c_1 = \underline{1/2}$

$-2 = c_1 + c_2 \qquad c_2 = -2 - 1/2 = \underline{-5/2}$

$$Y_{(x)} = \frac{1}{2} e^{-2x} \begin{bmatrix} 2\cos x \\ \cos x - \sin x \end{bmatrix} - \frac{5}{2} e^{-2x} \begin{bmatrix} 2\sin x \\ \cos x + \sin x \end{bmatrix}$$

Qu 3. *Trouver la solution série de l'équation différentielle.*
Find the series solution of the differential equation.

$$y'' - xy' - y = 0.$$

$y_p = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6$

$y'_p = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5$

$$\begin{cases} y''_p = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 \\ -xy' = -a_1 x - 2a_2 x^2 - 3a_3 x^3 - 4a_4 x^4 - 5a_5 x^5 - 6a_6 x^6 \\ -y = -a_0 - a_1 x - a_2 x^2 - a_3 x^3 - a_4 x^4 - a_5 x^5 - a_6 x^6 \end{cases}$$

x^0: $2a_2 - a_0$ $\quad 2a_2 = a_0 \quad a_2 = \frac{a_0}{2}$

x^1: $6a_3 - a_1 - a_1$ $\quad 6a_3 = 2a_1 \quad a_3 = \frac{a_1}{3}$

x^2: $12a_4 - 2a_2 - a_2$

x^3: $20a_5 - 3a_3 - a_3$

x^4: $30a_6 - 4a_4 - a_4$

$a_4 = \frac{a_2}{4}$ $\quad a_2 = \frac{a_0}{2}$

$a_4 = \frac{a_0}{8}$

$a_5 = \frac{a_3}{5}$ $\quad a_3 = \frac{a_1}{3}$

$a_5 = \frac{a_1}{15}$

$x^s : (s+1)(s+2)a_{s+2} - s a_s - a_s = 0$

$$a_{s+2} = \frac{\cancel{(s+1)}a_s}{\cancel{(s+1)}(s+2)} = \frac{a_s}{s+2}$$

$$\boxed{a_{s+2} = \frac{a_s}{s+2}}$$

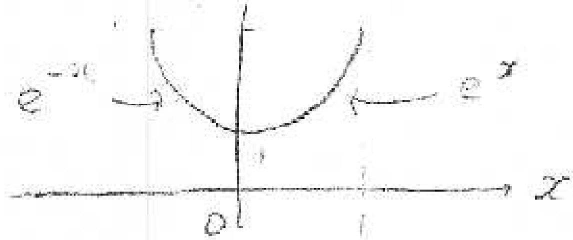
$$y_p = \overset{a_0}{a_0} + \overset{a_1}{a_1 x} + \overset{a_2}{\frac{a_0}{2}x^2} + \overset{a_3}{\frac{a_1}{3}x^3} + \overset{a_4}{\frac{a_0}{8}x^4} + \overset{a_5}{\frac{a_1}{15}x^5}$$

avec a_0 et a_1 indéfinis

Qu. 4. *Trouver les trois 1ers coefficients du développement de Fourier–Legendre de $f(x)$.*
Find the first three coefficients of the Fourier–Legendre expansion of the function $f(x)$.

$$f(x) = \begin{cases} e^{-x} & -1 < x < 0, \\ e^{x} & 0 < x < 1. \end{cases}$$

fonction paire



$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2} \times 2\int_0^1 e^x = e^x \Big|_0^1$$

$$= e - 1 = 1.7183$$

$$a_1 = \frac{3}{2}\int_{-1}^{1} f(x)\,x\,dx = 0 \quad | \text{ c.q } f(x)\,x \text{ est impaire}$$

$$a_2 = \frac{5}{2}\int_{-1}^{1} f(x)\,P_2(x)\,dx = 5\int_0^1 e^x \frac{1}{2}(3x^2 - 1)\,dx$$

$$= \frac{5}{2}\left[-\int_0^1 e^x\,dx + 3\int_0^1 x^2 e^x\,dx\right]$$

$$\int_0^1 x^2 e^x\,dx = x^2 e^x \Big|_0^1 - 2\int_0^1 x e^x\,dx$$

$$= e - 2\left[x e^x \Big|_0^1 - \int_0^1 e^x\,dx\right]$$

$$= e - 2e + 2(e - 1)$$

$$= e - 2$$

$$\Rightarrow a_2 = \frac{5}{2}\left[-(e-1) + 3(e-2)\right]$$

$$= \frac{5}{2}(1 - e + 3e - 6) = \frac{5}{2}(2e - 5)$$

$$= 1{,}09141$$

Qu. 5(a) *Évaluer l'intégrale par la quadrature gaussienne à 3 points.*

Evaluate the integral by Gauss' three-point quadrature.

$$I = \int_{0.2}^{1.8} \cos x^2 \, dx.$$

Attention : *Les angles en radians.* / Angles in radians.

$$\int_a^b \longrightarrow \int_{-1}^1 \text{ avec / with } x = \frac{b-a}{2}t + \frac{b+a}{2}.$$

$$dx = \frac{1.8-0.2}{2}dt = 0.8\,dt$$

$$I = 0.8\int_{-1}^{1} \cos(0.8t+1)^2 \, dt$$

$$= 0.8\left[\frac{5}{9}\cos\left(-0.8\sqrt{\tfrac{3}{5}}+1\right)^2 + \frac{8}{9}\cos 1 + \frac{5}{9}\cos\left(0.8\sqrt{3/5}+1\right)^2\right]$$

$$= 0.437932$$

Qu. 5(b) *Trouver le rayon de convergence de la série.*

Find the radius of convergence of the series.

$$\sum_{n=2}^{\infty} \frac{\ln n}{n} x^n.$$

$$\frac{1}{R} = \lim_{n\to\infty}\left(\frac{\ln(n+1)}{n+1}\,\frac{n}{\ln n}\right) = \lim_{n\to\infty}\frac{n}{n+1}\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}$$

$$= \lim_{n\to\infty}\frac{n}{n}\,\frac{1}{1+n}\lim_{n\to\infty}\frac{n}{n+1} = 1$$

par l'Hôpital

$$R = 1$$

$$\lim_{x\to\infty}\frac{\ln(x+1)}{\ln x} = \lim \frac{x}{x+1} = 1$$

$\mathcal{L}(\cosh) = \frac{s}{s^2-a^2}$

Qu. 6(a). *Trouver les transformées de Laplace de $f(t)$.*
Find the Laplace transforms of $f(t)$.

$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$

$$f(t) = 3\cosh 2t + 4\sin 5t.$$

$$F(s) = \frac{3s}{s^2-4} + \frac{20}{s^2+25}$$

$$f(t) = e^{-2t}\left(t^3 + \cos 3t\right).$$

$\mathcal{L}(t^3) = \frac{3!}{s^3}$

$$F(s) = \frac{3!}{(s+2)^3} + \frac{s+2}{(s+2)^2+9}$$

$\mathcal{L}(\cos 3t) = \frac{s}{s^2+9}$

Qu. 6(b). *Trouver les transformées de Laplace inverses de $F(s)$.*
Find the inverse Laplace transforms of $F(s)$.

$$F(s) = \frac{4(s+1)}{s^2-16} = \frac{4(s+1)}{(s+4)(s-4)} = \frac{A}{s+4} + \frac{B}{s-4}$$

$= As - 4A + Bs + 4B = 4s + 4$

10/10 $f(t) = \frac{3}{2}e^{-4t} + \frac{5}{2}e^{4t}$

$4 = A + B \qquad A = 4 - B$

$4 = -4A + 4B$

$4 = -4(4-B) + 4B$

$20 = 8B \qquad B = \frac{20}{8} = 5/2$

$A = 4 - \frac{20}{8} = 3/2$

$$\cancel{F(s)} = \frac{1+e^{-2s}}{s+2} = \frac{e^{0s}+e^{-2s}}{(s-(-2))}$$

$$f(t) = e^{-2t}\frac{\ln - s}{\ln s + 2}$$

$e^{(s-2)} - e^{(s-2)}$

Q~~n~~. 7. *Résoudre le système donné par décomposition de Cholesky $GG^T\mathbf{x} = \mathbf{b}$.*
Solve the following system by Cholesky decomposition $GG^T\mathbf{x} = \mathbf{b}$.

$$\begin{array}{rcrcrcr} -1 & & -2 & & +2 & & \\ x_1 & - & x_2 & - & 2x_3 & = & -1 \\ -x_1 & + & 5x_2 & + & 4x_3 & = & 7 \\ -2x_1 & + & 4x_2 & + & 14x_3 & = & -4 \end{array}$$

$$\begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} g_{11} & g_{21} & g_{31} \\ 0 & g_{22} & g_{32} \\ 0 & 0 & g_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 5 & 4 \\ -2 & 4 & 14 \end{bmatrix}$$

$g_{11}^2 = 1 \quad g_{11} = 1 \ \underline{\geq 0}$

$g_{11}g_{21} = -1 \quad g_{21} = -1$

$g_{11}g_{31} = -2 \quad g_{31} = -2$

$g_{21}^2 + g_{22}^2 = 5 \quad g_{22} = 2 \ \underline{\geq 0}$

$g_{21}g_{31} + g_{22}g_{32} = 4 \quad g_{32} = 1$

$g_{31}^2 + g_{32}^2 + g_{33}^2 = 14 \quad g_{33} = 3 \ \underline{\geq}$

$$G = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & 1 & 3 \end{bmatrix}$$

$$y = \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix}$$

$$x = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

✓

$Gy = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -4 \end{bmatrix}$$

$y_1 = -1$

$-y_1 + 2y_2 = 7 \quad y_2 = 3$

$-2y_1 + y_2 + 3y_3 = -4 \quad y_3 = 3$

$2 + 3 + 3y_3 = -4$

$y_3 = \frac{-9}{3} = 3$

$G^T x = y$

$$\begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix}$$

$3x_3 = -3 \quad x_3 = -1$

$2x_2 + x_3 = 3 \quad x_2 = 2$

$\overset{-2}{x_1 - x_2} \overset{+2}{- 2x_3} = -1 \quad x_1 = -1$