NOM/NAME: SOLUTIONS

N° d'ÉT/ST.#:

Université d'Ottawa · University of Ottawa

1/8

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Test mi-session 1
Durée: 80 min
Place: MRT 221
14 février 2007
17:30–18:50

Prof.: Rémi Vaillancourt

MAT 2784 B

Midterm 1
Time: 80 min
Place: MRT 221
14th of February 2007
17:30–18:50

Instructions:

(a) *À livre fermé. Tout type de calculatrices autorisé.*
Closed book. All types of calculators are allowed.

(b) *Répondre sur le questionnaire. Réponses numériques dans les boîtes.*
Answer on the question sheets. Fill-in boxes with numerical answers.

(c) *Les questions sont d'égale valeur. Répondre à 6 des 7 questions ; seules les 6 meilleures réponses seront comptées*
All questions have the same value. Answer 6 out of 7 questions; only the 6 best answers will count.

(d) *Donner le détail de vos calculs.*
Show all computation.

(e) *Une feuille couleur de tables sera distribuée.*
A one-page table on colored paper will be distributed.

(f) *Tous les angles sont en RADIANS. Tester et ajuster votre calculatrice.*
All angles are in RADIAN measures. Test and adjust your calculators.

$$\sin 1.123456789 = 0.90160112364453$$

L'équation différentielle homogène du 1er ordre,
The first order homogeneous differential equation,

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

admet un facteur d'intégration
admits an integrating factor

$$\mu(x) \quad \text{ou/or} \quad \mu(y)$$

selon que
according to

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x) \Longrightarrow \mu(x) = e^{\int f(x)\,dx},$$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y) \Longrightarrow \mu(y) = e^{-\int g(y)\,dy}.$$

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
TOTAL	/60

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Qu. 1. *Trouver un facteur d'intégration, rendre l'équation différentielle exacte et résoudre le problème à valeur initiale.*

Find an integration factor, make the differential equation exact and solve the initial value problem.

$$(1 - x^2 y)\,dx + x^2(y - x)\,dy = 0, \qquad y(1) = 2.$$

$\Leftrightarrow \quad M\,dx + N\,dy = 0$

$M_y = -x^2 \qquad N_x = 2xy - 3x^2 \qquad M_y \neq N_x$ eq non exacte

$$f(x) = \frac{M_y - N_x}{N} = \frac{-x^2 - 2xy + 3x^2}{x^2y - x^3} = \frac{2x^2 - 2xy}{x^2y - x^3} = \frac{2(x^2 - xy)}{-x(x^2 - xy)} = -\frac{2}{x}$$

$$\mu(x) = e^{\int f(x)\,dx} = e^{\int -\frac{2}{x}\,dy} = x^{-2}$$

$$du = x^{-2}(1 - x^2y)\,dx + x^{-2}(x^2y - x^3)\,dy = 0$$

$$\left(\frac{1}{x^2} - y\right)dx + (y - x)\,dy = 0$$

$$u(x,y) = \int N\,dy + T(x) = \int (y - x)\,dy + T(x)$$

$$= \frac{y^2}{2} - xy + T(x)$$

$$\frac{\partial u}{\partial x} = M \Leftrightarrow -y + T'(x) = M$$

$$= \frac{1}{x^2} - y \qquad \Rightarrow T'(x) = \frac{1}{x^2}$$

$$\Rightarrow T(x) = -\frac{1}{x}$$

Sol. gen $\quad u(x,y) = \frac{y^2}{2} - xy - \frac{1}{x}$

$u(x,y) = C \quad \Rightarrow \quad \frac{y^2}{2} - xy - \frac{1}{x} = C$

C.I $\quad y(1) = 2 \Rightarrow \frac{2^2}{2} - 2 - 1 = C \Rightarrow C = -1$

Sol. unique $\quad \boxed{\frac{y^2}{2} - xy - \frac{1}{x} = -1}$

Qu. 2. *Résoudre le problème à valeur initiale.*

Solve the initial value problem.

$$y' - y\cos x = \cos x, \qquad y(0) = 2.$$

(1) $\mu(x) = e^{\int \cos x \, dx} = e^{+\sin x}$

$(e^{+\sin x} y)' = \cos x \, e^{+\sin x}$

$e^{+\sin x} y = \int e^{+\sin x} \cos x \, dx + c$

$= +e^{+\sin x} + c$

$y(x) = +1 + c\, e^{-\sin x}$

$y(0) = +1 + c = 2 \Rightarrow c = 1$

Sol. unique :

$y(x) = -1 + e^{-\sin x}$

(2) $\dfrac{dy}{dx} = \cos x \,(1-y)$

$\int \dfrac{dy}{1-y} = \int \cos x \, dx + c$

$-\ln(1-y) = \sin x + c_1$

$e^{\ln(1-y)^{-1}} = e^{(\sin x + c_1)}$

$\dfrac{1}{1-y} = c\, e^{\sin x}$

$1-y = \dfrac{1}{c} e^{-\sin x}$

CI : $1-2 = \dfrac{1}{c} \Rightarrow c = -1$

$y = +1 + e^{-\sin x}$

Qu. 3. *Résoudre le problème aux valeurs initiales.*

Solve the initial value problem.

$$y'' + 4y = 6\sin x, \qquad y(0) = -2, \quad y'(0) = 1.$$

$\lambda^2 + 4 = 0$

$\lambda = \sqrt{-4} = 0 \pm 2i$

$\hookrightarrow y_h = C_1 \cos(2x) + C_2 \sin(2x)$

$4y_p = 4A\cos(x) + 4B\sin(x)$

$y_p' = -A\sin(x) + B\cos(x)$

$y_p'' = -A\cos(x) - B\sin(x)$

$y'' + 4y = \cos(x)(4A - A) + \sin(x)(4B - B) = 6\sin(x)$

$3A\cos(x) + 3B\sin(x) = 6\sin(x)$

$A = 0 \quad B = 2$

$y = y_h + y_p = C_1\cos(2x) + C_2\sin(2x) + 2\sin(x)$

$y(0) = C_1 = -2$

$y'(0) = -2C_1\sin(2x) + 2C_2\cos(2x) + 2\cos(x)\Big|_{x=0}$

$= 2C_2 + 2 = 1$

$C_2 = -1/2$

$$\boxed{y = -2\cos(2x) - \tfrac{1}{2}\sin(2x) + 2\sin(x)}$$

Qu. 4. *Résoudre le problème aux valeurs initiales*

Solve the initial value problem.

$$x^2y'' + 5xy' + 3y = 0, \qquad x > 0, \quad y(1) = 2, \quad y'(1) = -5.$$

Posons $y = x^m$ $\quad y' = m x^{m-1}$ $\quad y'' = m(m-1)x^{m-2}$

L'eq. devient

$$m(m-1)x^m + 5mx^m + 3x^m = 0$$

$$\Leftrightarrow m(m-1) + 5m + 3 = 0$$

$$\Leftrightarrow m^2 - m + 5m + 3 = 0$$

$$\Leftrightarrow m^2 + 4m + 3 = 0 \quad \Rightarrow m_1 = -3 \text{ et } m_2 = -1$$

Sol. gén. $\boxed{y(x) = c_1 x^{-3} + c_2 x^{-1}}$

$$y'(x) = -3c_1 x^{-4} - c_2 x^{-2}$$

$y(1) = 2 \Rightarrow c_1 + c_2 = 2 \quad \Rightarrow c_1 = 2 - c_2$ ①

$y'(1) = -5 \Rightarrow -3c_1 - c_2 = -5$ ②

① ds ② $\Rightarrow -3(2 - c_2) - c_2 = -5$

$\Rightarrow -6 + 3c_2 - c_2 = -5$

$\Rightarrow 2c_2 = 1 \Rightarrow c_2 = \frac{1}{2}$

$\Rightarrow c_1 = \frac{3}{2}$

Sol. unique

$$\boxed{y(x) = \frac{3}{2}x^{-3} + \frac{1}{2}x^{-1}}$$

Qu. 5. *Trouver la solution générale.*

Find the general solution.

$$y'' + 6y' + 9y = \frac{e^{-3x}}{x^3}.$$

$Ly = y'' + 6y' + 9y = 0$

$\Rightarrow \lambda^2 + 6\lambda + 9 = 0$

$\Rightarrow \lambda_{1,2} = -3$

$y_1(x) = e^{-3x}$ $\qquad y_2(x) = u(x)\, y_1(x) = x e^{-3x}$

$$\boxed{y_g(x) = A e^{-3x} + B x e^{-3x}}$$

$y_p(x) = C_1(x) e^{-3x} + C_2(x) x e^{-3x}$

$$\begin{vmatrix} e^{-3x} & x e^{-3x} \\ -3e^{-3x} & (1-3x)e^{-3x} \end{vmatrix} \begin{vmatrix} C_1' \\ C_2' \end{vmatrix} = \begin{vmatrix} 0 \\ e^{-3x}/x^3 \end{vmatrix} \quad L_2 + 3L_1$$

$$\Leftrightarrow \begin{vmatrix} e^{-3x} & x e^{-3x} \\ 0 & e^{-3x} \end{vmatrix} \begin{vmatrix} C_1' \\ C_2' \end{vmatrix} = \begin{vmatrix} 0 \\ \frac{e^{-3x}}{x^3} \end{vmatrix}$$

$e^{-3x} C_2' = \frac{e^{-3x}}{x^3} \Rightarrow C_2' = \frac{e^{-3x}}{x^3} \cdot e^{3x} = \frac{1}{x^3} \Rightarrow C_2' = \frac{1}{x^3}$

$\Rightarrow \boxed{C_2 = \frac{x^{-2}}{-2}}$

$e^{-3x} C_1 + x e^{-3x} C_2' = 0 \Rightarrow C_1' = -x e^{-3x} C_2' \cdot e^{3x}$

$= -x C_2'$ $\quad$ or $C_2' = \frac{1}{x^3}$

$\Rightarrow C_1' = -\frac{1}{x^2}$

$\Rightarrow \boxed{C_1 = \frac{1}{x}}$

$$\boxed{y_p(x) = \frac{1}{x} e^{-3x} - \frac{x^{-2}}{2} x e^{-3x}}$$

sol. gen.

$$\boxed{y_g(x) = A e^{-3x} + B x e^{-3x} + \frac{1}{x} e^{-3x} - \frac{x^{-1}}{2} e^{-3x}}$$

$$\left[\frac{1}{x} - \frac{1}{2}\frac{1}{x}\right] e^{-3x} = \frac{1}{2}\frac{1}{x} e^{-3x}$$

Qu. 6. *Compléter le tableau pour la récurrence de point fixe :*
Fill the boxes for the fixed point iteration:

$$x_{n+1} = \sqrt{2x_n + 3} \equiv g(x_n).$$

n	x_n	Δx_n	$\Delta^2 x_n$
1	$x_1 =$ 4.0000		
		-0,6834	
2	$x_2 =$ $\sqrt{11} \approx$ 3,3166		0,4705
		-0,2129	
3	$x_3 =$ 3,1037		

Accélérer la convergence par la méthode d'Aitken.
Accelerate the convergence by the Aitken process.

$$a_1 = x_1 - \frac{(\Delta x_1)^2}{\Delta^2 x_1} = \boxed{3,0074}$$

$4 - \frac{(0,6834)^2}{0,4705}$

Accélérer la convergence par la méthode de Steffensen
Accelerate the convergence by the Steffenseen process.

$$s_1 = a_1,$$

$$z_1 = g(s_1) = \boxed{3,0025}$$

$$z_2 = g(z_1) = \boxed{3,0008}$$

$$s_2 = s_1 - \frac{(z_1 - s_1)^2}{z_2 - 2z_1 + s_1} = \boxed{2,9999}$$

$$3,0074 - \frac{(3,0025 - 3,0074)^2}{3,0008 - 2 \cdot 3,0025 + 3,0074}$$

Qu. 7. *Compléter le tableau de différences divisées :*
Complete the divided difference table:

i	x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	3.2	22.0			
			8.400		
1	2.7	17.8		2.856	
			$\boxed{2.1176}$		
					−0.528
2	1.0	14.2		$\boxed{2.0116}$	
			$\boxed{6.3421}$		$\boxed{-4.2735}$
3	4.8	38.3		$\boxed{-10.3814}$	
			$\boxed{-41.4125}$		
4	5.6	5.17			

Construire le polynôme de degré 3 qui interpole les données aux 4 points de $x_0 = 3.2$ à $x_3 = 4.8$.
Construct the cubic interpolating polynomial which interpolates the data at the 4 points $x_0 = 3.2$ to $x_3 = 4.8$.

$$y_3(x) = f[x_0] + (x - x_0)\, f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\, f[x_0, x_1, x_2, x_3]$$

$$= 22.0 + (x - 3.2)\, 8.400 + (x - 3.2)(x - 2.7)\, 2.856 + (x - 3.2)(x - 2.7)(x - 1)(-0.528)$$