MAT 2784B   07.04.09   D9.1

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6.50 $y''+4y'+3y = \begin{cases} 4e^{1-t} & 0 \le t < 1 \\ 4 & t \ge 1 \end{cases}$ $y(0)=0$, $y'(0)=0$

$y''+4y'+3y = g(t)$

$s^2 Y(s) - \cancel{y(0)} - \cancel{sy'(0)} + 4s Y(s) - \cancel{4y(0)} + 3Y(s) = g(s)$

$s^2 Y(s) + 4s Y(s) + 3Y(s) = g(s)$

$Y(s)\,(s^2+4s+3) = g(s)$

$f(t)$	$F(s)$
$e^{at}f(t)$	$F(s-a)$
e^{at}	$\frac{1}{s-a}$
$u(t-a)$	$e^{-as}\frac{1}{s}$
t^n	$n!/s^{n+1}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$

$g(t) = 4e^{1-t} - u(t-1)\,4e^{1-t} + 4u(t-1)$

$= 4ee^{-t} - 4e^{-(t-1)}u(t-1) + 4u(t-1)$

$g(s) = \frac{4e}{s+1} - 4e^{-s}\cdot\frac{1}{s+1} + \frac{4e^{-s}}{s}$

$g(s) = \frac{4e}{s+1} - \frac{4e^{-s}}{s+1} + \frac{4e^{-s}}{s}$

$Y(s)\,(s^2+4s+3) = \frac{4e}{s+1} - \frac{4e^{-s}}{s+1} + \frac{4e^{-s}}{s}$

$Y(s) = \frac{4e}{s+1} - \frac{4e^{-s}}{s+1} + \frac{4e^{-s}}{s}\left(\frac{1}{(s+1)(s+3)}\right)$

$Y(s) = \frac{4e}{(s+1)^2(s+3)} - \frac{4e^{-s}}{(s+1)^2(s+3)} + \frac{4e^{-s}}{s(s+1)(s+3)}$

Fractions partielles

$\frac{1}{(s+1)^2(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+3)}$

$= A(s+1)(s+3) + B(s+3) + C(s+1)^2$

$= A(s^2+4s+3) + Bs + 3B + C(s^2+2s+1)$

$= As^2 + 4sA + 3A + Bs + 3B + Cs^2 + 2Cs + 1C$

$= (A+C)s^2 + (4A+B+2C)s + 3A+3B+1C$

$A+C=0$ $4A+B+2C=0$ $3A+3B+1C=1$

$A=-C$ $A=-1/4$ $B=\frac{1}{2}$ $C=1/4$

$\frac{1}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+3)}$

$= A(s+1)(s+3) + B(s)(s+3) + Cs(s+1)$

✓

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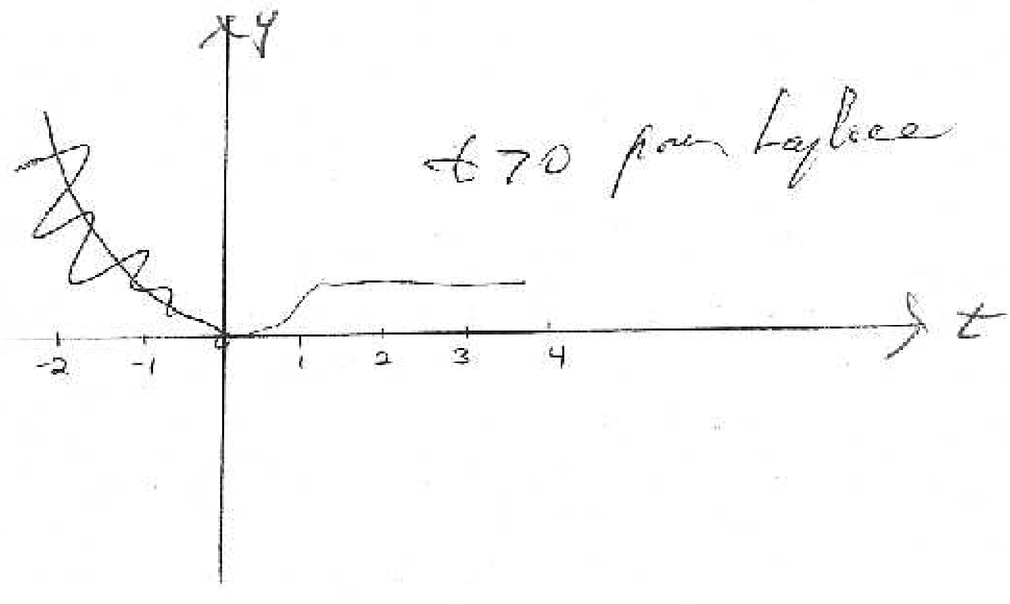
$$1 = As^2 + As + 3As + 3A + Bs^2 + 3Bs + Cs^2 + Cs$$
$$= (A+B+C)s^2 + (4A+3B+C)s + 3A$$
$$3A = 1$$
$$A = \tfrac{1}{3} \qquad B = -\tfrac{1}{2} \qquad C = \tfrac{1}{6}$$

$$Y(s) = 4e\left(-\frac{1}{4(s+1)} + \frac{1}{2(s+1)^2} + \frac{1}{4(s+3)}\right)$$
$$= 4e^{-s}\left(-\frac{1}{4(s+1)} + \frac{1}{2(s+1)^2} + \frac{1}{4(s+3)}\right)$$
$$+ 4e^{-s}\left(\frac{1}{3s} - \frac{1}{2(s+1)} + \frac{1}{6(s+3)}\right)$$

$$y(t) = 4e\left(-\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} + \frac{1}{4}e^{-3t}\right)$$
$$-4\left(-\frac{1}{4}e^{-(t-1)} + \frac{1}{2}(t-1)e^{-(t-1)} + \frac{1}{4}e^{-3(t-1)}\right)u(t-1)$$
$$+4\left(\frac{1}{3} - \frac{1}{2}e^{-(t-1)} + \frac{1}{6}e^{-3(t-1)}\right)u(t-1)$$



6.53 $y''-y = \sin(t) + \delta(t-\pi/2)$ $y(0)=3,5$ $y'(0)=-3,5$

$\mathcal{L}(y)=Y$ $\mathcal{L}(y'')= s^2Y - sy(0)-y'(0) = s^2Y - 3,5s+3,5$

$\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$, $\mathcal{L}(\delta(t-\pi/2)) = e^{-\pi/2 \cdot s}$

$\mathcal{L}(y''-y) = s^2Y-3,5s+3,5-Y = (s^2-1)Y-3,5(s-1)$

$Y(s^2-1) = \frac{1}{s^2+1} + e^{-\pi/2\cdot s} + 3,5(s-1)$

$Y = \frac{1}{s^4-1} + e^{-\pi/2\cdot s}\frac{1}{s^2-1} + 3,5\cdot\frac{1}{s+1}$ V. table.

$y(t) = \underbrace{1/2 \sinh(t) - 1/2 \sin(t)} + u(t-\pi/2)\sinh(t-\pi/2) + 3,5e^{-t}$

SOLUTION RAPIDE

Zhi'a

A.D.
D9.4

<u>6,53</u> $y'' - y = \sin t + \delta(t - \frac{\pi}{2})$ $\quad y(0) = 3{,}5$, $y'(0) = -3{,}5$

$$s^2 Y(s) - s\underset{3,5}{y(0)} - \underset{-3,5}{y'(0)} - Y(s) = g(s)$$

$$s^2 Y(s) - 3{,}5s + 3{,}5 - Y(s) = g(s)$$

$$(s^2 - 1)Y(s) = +3{,}5s - 3{,}5 + g(s)$$

$$g(t) = \sin t + \delta(t - \pi/2)$$

$f(t)$	$F(s)$
$\frac{1}{\omega}\sin\omega t$	$\frac{1}{s^2+\omega^2}$
$\delta(t-a)$	e^{-as}

$$g(s) = \frac{1}{s^2+1} + e^{-\frac{\pi}{2}s}$$

$$(s^2-1)Y(s) = +3{,}5s - 3{,}5 + \frac{1}{s^2+1} + e^{-\pi/2 s}$$

$$Y(s) = \left[3{,}5s - 3{,}5 + \frac{1}{s^2+1} + e^{-\pi/2 s}\right]\left(\frac{1}{s^2-1}\right)$$

$$= \frac{3{,}5s}{s^2-1} - \frac{3{,}5}{s^2-1} + \frac{1}{(s^2+1)(s^2-1)} + \frac{e^{-\pi/2 s}}{s^2-1}$$

Fraction ~~partielle~~ simple :

$$\frac{1}{\underbrace{(s^2+1)(s^2-1)}_{s^4-1}} = \frac{1}{(s^2+1)(s+1)(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{(s+1)} + \frac{D}{(s-1)}$$

$$= (As+B)(s^2-1) + C(s^2+1)(s-1) + D(s^2+1)(s+1)$$

$$= As^3 + Bs^2 - As - B + C(s^3 + 1s - 1s^2 - 1) + D(s^3 + 1s + 1s^2 + 1)$$

$$= As^3 + Bs^2 - As - B + Cs^3 + Cs - Cs^2 - C + Ds^3 + Ds + Ds^2 + D$$

$$= (A + C + D)s^3 + (B - C + D)s^2 + (-A + C + D)s + (-B - C + D)$$

$\boxed{\text{paire} \Rightarrow A = 0}$ $\quad A = 0 \quad B = -1/2 \quad C = -1/4 \quad D = 1/4$

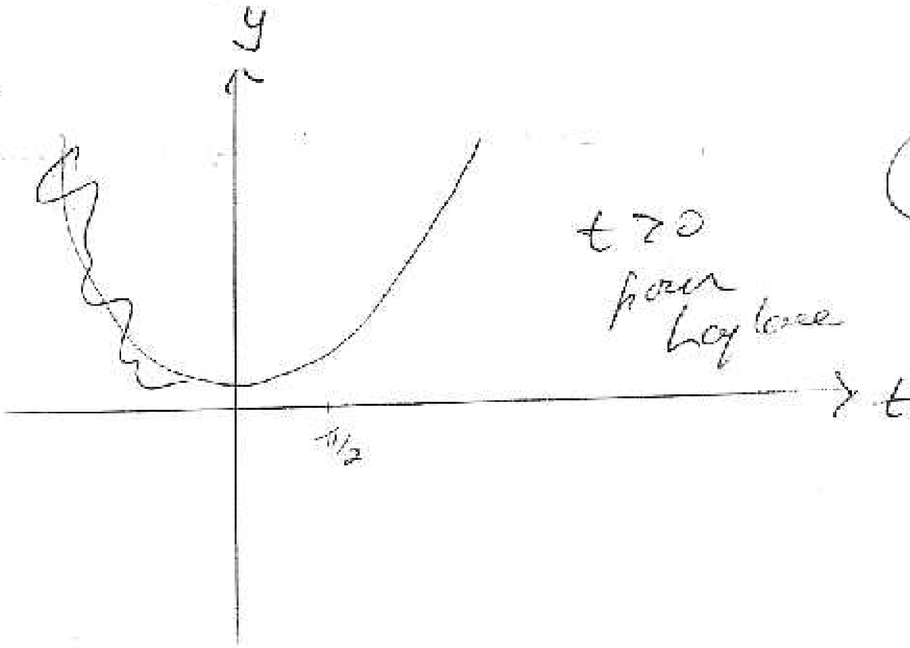
$$Y(s) = \frac{3{,}5\,s}{s^2-1} - \frac{3{,}5}{s^2-1} - \frac{1}{2(s^2+1)} - \frac{1}{4(s+1)} + \frac{1}{4(s-1)} + \frac{e^{-\pi/2 s}}{(s^2-1)}$$

$$y(t) = 3{,}5\cosh t - 3{,}5\sinh t - \frac{1}{2}\sin t - \frac{1}{4}e^{-t} + \frac{1}{4}e^{t} + \sinh(t - \pi/2)\,u(t - \pi/2)$$

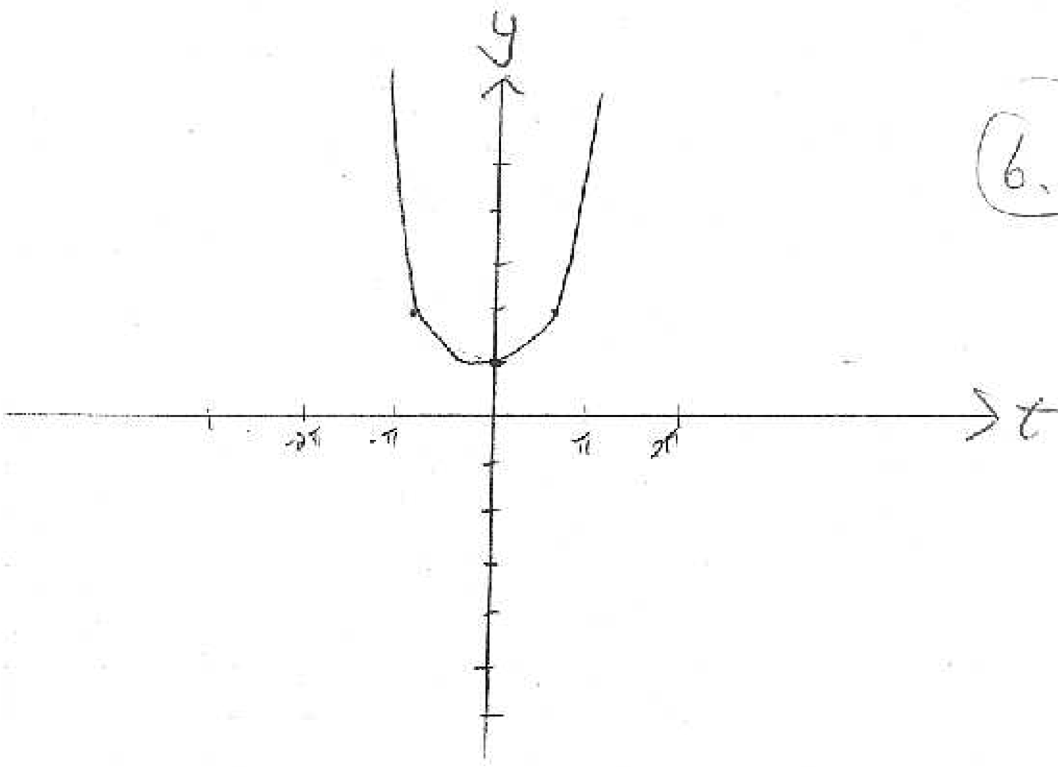
$\rightarrow$

3

D9.5



6.53



6.58

<u>6.58</u> $y(t) = t + e^t + \int_0^t y(\tau)\cosh(t-\tau)\,d\tau$

$y(t) = t + e^t + y * \cosh(t)$

$Y(s) = \frac{1}{s^2} + \frac{1}{s-1} + Y(s)\left(\frac{s}{s^2-1}\right)$

$Y(s) - Y(s)\left(\frac{s}{s^2-1}\right) = \frac{1}{s^2} + \frac{1}{s-1}$

$\frac{(s^2-s-1)}{s^2-1} Y(s) = \frac{1}{s^2} + \frac{1}{s-1}$

$Y(s) = \left(\frac{1}{s^2} + \frac{1}{(s-1)}\right)\left(\frac{(s-1)(s+1)}{s^2-s-1}\right)$

$Y(s) = \frac{s^2-1}{s^2(s^2-s-1)} + \frac{(s+1)}{(s^2-s-1)}$

Fraction partielle

$\frac{s^2-1}{s^2(s^2-s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2-s-1}$

$= As(s^2-s-1) + B(s^2-s-1) + s^2(Cs+D)$

$s^2-1 = As^3 - As^2 - As + Bs^2 - Bs - B + Cs^3 + Ds^2$

$s^2-1 = (A+C)s^3 + (-A+B+D)s^2 + (-A-B)s - B$

$-B = -1 \qquad -A-1 = 0 \qquad -1+C=0 \qquad D=-1$

$B = 1 \qquad A = -1 \qquad C = 1$

$Y(s) = \frac{-1}{s} + \frac{1}{s^2} + \frac{s-1}{s^2-s-1} + \frac{s+1}{s^2-s-1}$

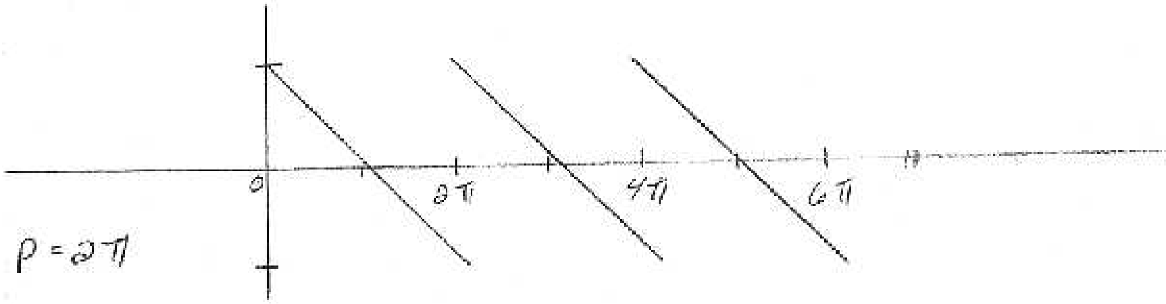
$Y(s) = -\frac{1}{s} + \frac{1}{s^2} + \frac{2s}{s^2-s-1}$

$\frac{s}{s^2-s-1} = \frac{(s-\frac{1}{2})+\frac{1}{2}}{(s^2-s+\frac{1}{4})-\frac{5}{4}} = \frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2-\frac{5}{4}} + \frac{1}{2}\frac{1}{\sqrt{5/4}}\cdot\frac{\sqrt{5/4}}{(s-\frac{1}{2})^2-\frac{5}{4}}$

$Y(s) = -\frac{1}{s} + \frac{1}{s^2} + 2\frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2-\frac{5}{4}} + \frac{1}{\sqrt{5/4}}\cdot\frac{\sqrt{5/4}}{(s-\frac{1}{2})^2-\frac{5}{4}}$ 10

5 $y(t) = -1 + t + 2e^{-t/2}\cosh(\sqrt{5/4}\,t) + \frac{1}{\sqrt{5/4}}\sinh(\sqrt{5/4}\,t)$ →

6.60 $f(t) = \pi - t \quad 0 < t < 2\pi$



$P = 2\pi$

$$F(s) = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\, dt$$

$$F(s) = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} (\pi - t)\, dt$$

intégrale par parties

$uv - \int v\,du \quad u = (\pi - t) \quad du = e^{-st}dt$
$du = -1 \quad v = \frac{-1}{s}e^{-st}$

$$F(s) = \frac{1}{1-e^{-2\pi s}} \left((\pi - t)\left(-\frac{e^{-st}}{s}\right)\Big|_0^{2\pi} - \int_0^{2\pi} -\frac{1}{s} e^{-st}(-1)\, dt \right)$$

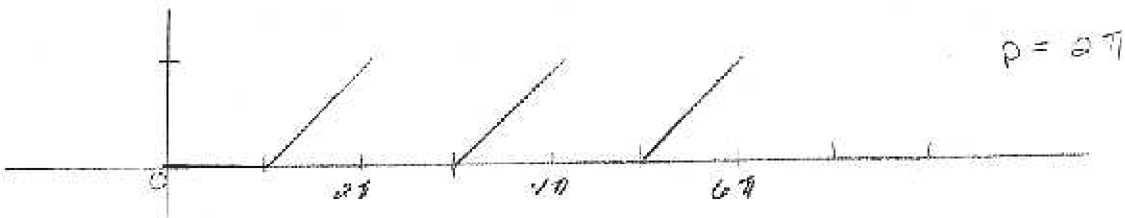
$$F(s) = \frac{1}{1-e^{-2\pi s}} \left(-\frac{(\pi - t)e^{-st}}{s}\Big|_0^{2\pi} + \frac{1}{s}\int_0^{2\pi} e^{-st}\, dt \right)$$

$$F(s) = \frac{1}{1-e^{-2\pi s}} \left(-\frac{\pi}{s} + \frac{\pi e^{-2\pi s}}{s} - \frac{1}{s^2} e^{-st}\Big|_0^{2\pi} \right)$$

$$F(s) = \frac{1}{1-e^{-2\pi s}} \left(\frac{-\pi}{s} + \frac{\pi e^{-2\pi s}}{s} - \frac{1}{s^2} - \frac{e^{-2\pi s}}{s^2} \right)$$

6.64

$$f(t) = \begin{cases} 0 & \text{si } 0 < t < \pi \\ t-\pi & \text{si } \pi < t < 2\pi \end{cases}$$



$$F(s) = \cancel{\frac{1}{1-e^{-2\pi s}} \int_0^{\pi} e^{-st}(0)\,dt} + \frac{1}{1-e^{-2\pi s}} \int_{\pi}^{2\pi} e^{-st}(t-\pi)\,dt$$

$$F(s) = \frac{1}{1-e^{-2\pi s}} \int_{\pi}^{2\pi} e^{-st}(t-\pi)\,dt$$

intégrale par parties $\int u\,dv = uv - \int v\,du$ $\quad \left[\begin{array}{ll} u = (t-\pi) & dv = e^{-st}\,dt \\ du = 1\,dt & v = -\frac{1}{s} e^{-st} \end{array} \right.$

$$F(s) = \frac{1}{1-e^{-2\pi s}} \left(\frac{-1}{s}(t-\pi)e^{-st} \Big|_{\pi}^{2\pi} - \int_{\pi}^{2\pi} -\frac{1}{s} e^{-st}\,dt \right)$$

$$F(s) = \frac{1}{1-e^{-2\pi s}} \left(\frac{-(t-\pi)e^{-st}}{s} \Big|_{\pi}^{2\pi} + \frac{1}{s^2} e^{-st} \Big|_{\pi}^{2\pi} \right)$$

$$F(s) = \frac{1}{1-e^{-2\pi s}} \left(-\frac{\pi e^{-2\pi s}}{s} + \frac{1}{s^2} e^{-\pi s} - \frac{1}{s^2} e^{-2\pi s} \right)$$

12,13 $y' = x + \cos y$ $y(0) = 0$ $0 \le x \le 1$

Runge-Kutta d'ordre 4 avec $h = 0,1$

tracer solution

$y' = x + \cos y$ $f(x,y) = x + \cos y$

$x_0 = 0$ $y_0 = 0$ $h = 0,1$

n=0

$k_1 = h\,f(x_0, y_0) = 0,1 \times 1 = 0,1$

$k_2 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1) = 0,1 \times 1,048975 = 0,1048975$

$k_3 = h\,f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) = 0,1 \times 1,048625 = 0,1048625$

p.238 $k_4 = h\,f(x_0 + h, y_0 + k_3) = 0,1 \times 1,0445069 = 0,10445069$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0 + \frac{1}{6}(0,1 + 2(0,1048975) + 2(0,1048625) + 0,1044069)$

$= 0,104821$

n=1 $y = x + \cos y$ $x_1 = x_0 + h = 0,1$ $y_1 = 0,104821$

$k_1 = h\,f(x_1, y_1) = 0,1 \times [0,1 + \cos(0,104821)] = 0,10945$

$k_2 = h\,f(x_1 + h/2, y_1 + k_1/2) = 0,1 \times [0,1 + 0,1/2 + \cos(0,104821 + 0,10945/2)] = 0,11372995$

$k_3 = h\,f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2) = 0,1 \times [0,1 + 0,1/2 + \cos(0,104821 + 0,11373/2)] = 0,113696$

$k_4 = h\,f(x_1 + h, y_1 + k_3) = 0,1 \times [0,1 + 0,1 + \cos(0,104821 + 0,113696)] = 0,117622$

$y_2 = y_1 + 1/6\,(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0,104821 + 1/6\,(0,10945 + 2 \cdot 0,11373 + 2 \cdot 0,113696 + 0,117622)$

$y_2 = 0,218475$

n=2 $y = x + \cos y$ $x_2 = x_1 + h = 0,2$ $y_2 = 0,218475$

$k_1 = h\,f(x_2, y_2) = 0,1 \times [0,2 + \cos(0,218475)] = 0,1176229$

$k_2 = h\,f(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1) = 0,1 \times [0,25 + \cos(0,27728)] = 0,1211803$

$k_3 = h\,f(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2) = 0,1 \times [0,25 + \cos(0,279065)] = 0,121131$

$k_4 = h\,f(x_2 + h, y_2 + k_3) = 0,1 \times [0,3 + \cos(0,339606)] = 0,124289$

$y_3 = y_2 + 1/6\,(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0,218475 + 1/6\,(0,1176229 + 2 \cdot 0,1211803 + 2 \cdot 0,121131 + 0,124289)$

$= 0,339564$

8 →

D 9.10

$n = 3 \qquad y' = x + \cos y \qquad x_3 = 0,3 \qquad y_3 = 0,339564$

$k_1 = h f(x_3, y_3) = 0,1 \times [0,3 + \cos(0,339564)] = 0,124290$

$k_2 = h f(x_3 + \tfrac{1}{2}h, y_3 + \tfrac{1}{2}k_1) = 0,1 \times [0,35 + \cos(0,401709)] = 0,127039$

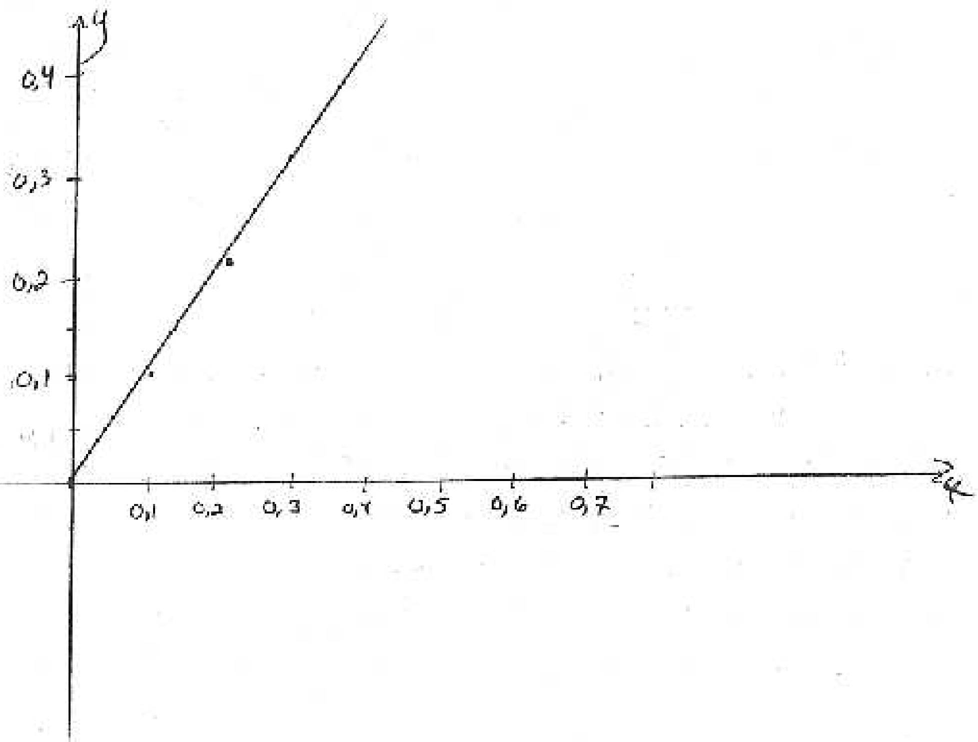
$k_3 = h f(x_3 + \tfrac{1}{2}h, y_3 + \tfrac{1}{2}k_2) = 0,1 \times [0,35 + \cos(0,4030835)] = 0,126986$

$k_4 = h f(x_3 + h, y_3 + k_3) = 0,1 \times [0,4 + \cos(0,46655)] = 0,129313$

$y_4 = y_3 + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0,339564 + \tfrac{1}{6}(0,124290 + 2 \cdot 0,127039 + 2 \cdot 0,126986 + 0,129313)$

$= 0,466506$



9

$\underline{12.17}$ $y' = x + 2\sin y$ $y(0) = 0$; $h = ~~1~~\ 0.1$

$f(x,y) = x + 2\sin y$ $x_0 = 0$ $y_0 = 0$ ODE23

<u>avec $n = 0$:</u>

$k_1 = 0{,}1\,(0 + 2\sin(0)) = 0{,}000000$
$k_2 = 0{,}1\,(0{,}05 + 2\sin(0)) = 0{,}005000$
$k_3 = 0{,}1\,(0{,}075 + 2\sin(0 + 0{,}003375)) = 0{,}008250$
$k_4 = 0{,}1\,(0{,}1 + 2\sin(0 + 0{,}001667 + 0{,}003667)] = 0{,}011067$

$y_1 = 0 + \frac{2}{9}(0) + \frac{1}{3}(0{,}005) + \frac{4}{9}(0{,}008250) = 0{,}005333$

$EE = -\frac{5}{72}(0) + \frac{1}{12}(0{,}005) + \frac{1}{9}(0{,}00825) - \frac{1}{8}(0{,}011067)$

$= -0{,}000050$

<u>avec $n = 1$ et $x_1 = 0{,}1$</u>

$k_1 = 0{,}1\,(0{,}1 + 2\sin(0{,}005333)) = 0{,}011067$
$k_2 = 0{,}1\,(0{,}15 + 2\sin(0{,}005333 + 0{,}005333)) = 0{,}017173$
$k_3 = 0{,}1\,(0{,}175 + 2\sin(0{,}005333 + 0{,}012880)) = 0{,}021142$
$k_4 = 0{,}1\,(0{,}2 + 2\sin(0{,}005333 + 0{,}002459 + 0{,}005724 + 0{,}009396))$
$= 0{,}024581$

$y_2 = 0{,}005333 + 0{,}002459 + 0{,}005724 + 0{,}009396 = 0{,}022912$

$EE = -\frac{5}{72}(0{,}011067) + \frac{1}{12}(0{,}017173) + \frac{1}{9}(0{,}021142) - \frac{1}{8}(0{,}024581)$

$= -0{,}000061$

6 décimales

12,22 $y' = x + \cos y$ $\quad y(0) = 0$

Adams-B-M ordre 3 $h = 0,1$ erreur pour $x = 0,5$

tracer la solution.

valeurs initiales

y_n^c	
$x_0 = 0$	$y_0 = 0$
$x_1 = 0,1$	$y_1 = 0,104821$
$x_2 = 0,2$	$y_2 = 0,218475$

$$y_{n+1}^p = y_n^c + \frac{h}{12}\left(23 f_n^c - 16 f_{n-1}^c + 5 f_{n-2}^c\right) \quad f_k^c = f(x_k, y_k^c)$$

$$y_{n+1}^c = y_n^c + \frac{h}{12}\left(5 f_{n+1}^p + 8 f_n^c - f_{n-1}^c\right) \quad f_k^p = f(x_k, y_k^p)$$

$n=2$

$$y_3^p = y_2^c + \frac{0,1}{12}\left(23 f_2^c - 16 f_1^c + 5 f_0^c\right)$$

$$= 0,218475 + 0,008333\left(23(1,1762) - 16(1,0945) + 5(1)\right)$$

$$y_3^p = 0,339651$$

$$y_3^c = y_2^c + \frac{0,1}{12}\left(5 f_3^p + 8 f_2^c - f_1^c\right)$$

$$= 0,218475 + 0,008333\left(5(1,2428) + 8(1,1762) - 1,0945\right)$$

$$y_3^c = 0,339556$$

$$\mathcal{E} \approx -\frac{1}{10}\left[y_{n+1}^c - y_{n+1}^p\right]$$

$$\mathcal{E} \approx -\frac{1}{10}\left[y_3^c - y_3^p\right] \approx -\frac{1}{10}\left(0,339556 - 0,339651\right) = 0,0000095$$

$$\mathcal{E} = 0,95 \times 10^{-5}$$

$n=3$

$$y_4^p = y_3^c + \frac{0,1}{12}\left(23 f_3^c - 16 f_2^c + 5 f_1^c\right)$$

$$= 0,339556 + 0,008333\left(23(1,2429) - 16(1,1762) + 5(1,0945)\right)$$

$$= 0,339556 + 0,008333(15,24)$$

$$y_4^p = 0,46655$$

$$y_4^c = y_3^c + \frac{0,1}{12}\left(5 f_4^p + 8 f_3^c - f_2^c\right)$$

$$= 0,339556 + 0,008333\left(5(1,2931) + 8(1,2429) - (1,1762)\right)$$

$$= 0,339556 + 0,008333(15,2325)$$

$$= 0,466489$$

$$E \approx -\frac{1}{10}\left[y_4^c - y_4^p\right] \approx -\frac{1}{10}\left(0,466489 - 0,46655\right) \approx 0,0000067$$

$$\mathcal{E} \approx 0,67 \times 10^{-5}$$

O 9.13