SOLUTIONS

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MAT2784
Devoir #6

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<u>4.2</u> $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ résoudre $y' = Ay$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix}$$

$$= (-1)(-4(2-\lambda)) + (2-\lambda)((2-\lambda)(2-\lambda))$$

$$= -4\lambda + 8 + (2-\lambda)(\lambda^2 - 4\lambda + 4) = -4\lambda + 8 + (2\lambda^2 - 8\lambda + 8 - \lambda^3 + 4\lambda^2 - 4$$
$$= -\lambda^3 + 6\lambda^2 - 16\lambda + 16$$

$$\begin{array}{l|l} -\lambda^3 + 6\lambda^2 - 16\lambda + 16 & \lambda - 2 \\ \hline -(-\lambda^3 + 2\lambda^2) & -\lambda^2 + 4\lambda - 8 \\ \quad 4\lambda^2 - 16\lambda + 16 & \\ \quad -(4\lambda^2 - 8\lambda) & \\ \qquad -8\lambda + 16 & \\ \qquad -8\lambda + 16 & \\ \qquad\quad 0 & \end{array}$$

$$\Rightarrow (\lambda - 2)(-\lambda^2 + 4\lambda - 8) = (\lambda - 2)(\lambda^2 - 4\lambda + 8)$$

$\lambda_1 = 2$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 32}}{2} = \frac{4 \pm \sqrt{-16}}{2}$$

$$= \frac{4 \pm 4i}{2} = 2 \pm 2i$$

$$\lambda_2 = 2 + 2i \qquad \lambda_3 = 2 - 2i$$

donc $\alpha = 2$, $\beta = 2$

premier vecteur propre ($\lambda_1 = 2$):

$[A - 2I]\vec{u} = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} u_1 = 0 \\ u_2 = t \\ u_3 = 0 \end{array}\right\} \quad \vec{u} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

deuxième vecteur propre ($\lambda_2 = 2 + 2i$):

$[A - (2+2i)I]\vec{v} = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} -2i & 0 & 4 & 0 \\ 0 & 2i & 0 & 0 \\ -1 & 0 & -2i & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 4i & 0 \\ 0 & 2 & 0 & 0 \\ -i & 0 & 2 & 0 \end{array}\right] \begin{array}{l} L_1(i) \\ L_2(i) \\ L_3(i) \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 2i & 0 \\ 0 & 1 & 0 & 0 \\ \cancel{1} & \cancel{0} & \cancel{2i} & \cancel{0} \end{array}\right]$$

$$\left.\begin{array}{l} v_1 = (-2i)t \\ V_2 = 0 \\ V_3 = t \end{array}\right\} \quad \vec{v} = t\begin{bmatrix} -2i \\ 0 \\ 1 \end{bmatrix} \quad \text{ou} \quad \vec{v} = t\begin{bmatrix} 2 \\ 0 \\ i \end{bmatrix}$$

troisième vecteur propre ($\lambda_3 = 2 - 2i$):

$[A - (2-2i)]\vec{w} = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} 2i & 0 & 4 & 0 \\ 0 & 2i & 0 & 0 \\ -1 & 0 & 2i & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 0 & 2i & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 2i & 0 \end{array}\right] \begin{array}{l} L_1(\frac{i}{2}) \\ L_2(\frac{i}{2}) \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & -2i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \vec{w} = t\begin{bmatrix} 2i \\ 0 \\ 1 \end{bmatrix} = t\begin{bmatrix} -2 \\ 0 \\ i \end{bmatrix}$$

alors $\vec{V} = \begin{bmatrix} 2 \\ 0 \\ i \end{bmatrix} e^{(2+2i)x} = \begin{bmatrix} 2 \\ 0 \\ i \end{bmatrix} e^{2x} e^{2ix}$

$$= \begin{bmatrix} 2 \\ 0 \\ i \end{bmatrix} e^{2x} (\cos 2x + i \sin 2x)$$

$$= e^{2x} \begin{bmatrix} 2(\cos 2x + i \sin 2x) \\ 0 \\ i(\cos 2x + i \sin 2x) \end{bmatrix} = \begin{bmatrix} 2\cos 2x + 2i \sin 2x \\ 0 \\ i\cos 2x - \sin 2x \end{bmatrix}$$

$$= e^{2x} \begin{bmatrix} 2\cos 2x \\ 0 \\ -\sin 2x \end{bmatrix} + i e^{2x} \begin{bmatrix} 2\sin 2x \\ 0 \\ \cos 2x \end{bmatrix}$$

Solution générale $y(x) = C_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} 2\cos 2x \\ 0 \\ -\sin 2x \end{bmatrix} e^{2x} + C_3 \begin{bmatrix} 2\sin 2x \\ 0 \\ \cos 2x \end{bmatrix} e^{2x}$

4.10 résoudre $y' = Ay + f(x)$

si $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ $\quad f(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & \sqrt{3} \\ \sqrt{3} & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) - (\sqrt{3})(\sqrt{3})$$

$$= (\lambda^2 - \lambda + \lambda - 1) - 3$$

$\lambda_{1,2} = \pm 2$ $\qquad = \lambda^2 - 4$

Vecteur propre pour $\lambda_1 = 2$:

$[A - 2I]\vec{u} = 0$

$$\Rightarrow \left[\begin{array}{cc|c} -1 & \sqrt{3} & 0 \\ \sqrt{3} & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\sqrt{3} & 0 \\ \sqrt{3} & -3 & 0 \end{array}\right]^{-L_1} \sim \left[\begin{array}{cc|c} 1 & -\sqrt{3} & 0 \\ 0 & 0 & 0 \end{array}\right]_{L_2 - \sqrt{3}L_1}$$

$u_1 = \sqrt{3}t$ $\quad \vec{u} = t\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$

$u_2 = t$

Vecteur propre pour $\lambda_2 = -2$:

$[A + 2I]\vec{v} = 0 \quad \Rightarrow \left[\begin{array}{cc|c} 3 & \sqrt{3} & 0 \\ \sqrt{3} & 1 & 0 \end{array}\right]$

$$\sim \left[\begin{array}{cc|c} 1 & \sqrt{3}/3 & 0 \\ \sqrt{3} & 1 & 0 \end{array}\right]^{L_1/3} \sim \left[\begin{array}{cc|c} 1 & \sqrt{3}/3 & 0 \\ 0 & 0 & 0 \end{array}\right]_{L_2 - \sqrt{3}L_1}$$

$V_1 = -\sqrt{3}/3$ $\quad \vec{v} = t\begin{bmatrix} -\sqrt{3}/3 \\ 1 \end{bmatrix} = t\begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$

$V_2 = t$

on obtient $y_h = C_1 \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} e^{-2x}$

pour trouver la solution particulière :

$y_p = \begin{bmatrix} a \\ b \end{bmatrix} \qquad y_p' = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad y' = Ay + f(x)$

$$\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & \sqrt{3} & -1 \\ \sqrt{3} & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \sqrt{3} & -1 \\ 0 & -4 & \sqrt{3} \end{array}\right]_{L_2 - \sqrt{3}L_1} \sim \left[\begin{array}{cc|c} 1 & \sqrt{3} & -1 \\ 0 & 1 & -\frac{\sqrt{3}}{4} \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & -1/4 \\ 0 & 1 & -\sqrt{3}/4 \end{array}\right]^{L_1 - \sqrt{3}L_2} \qquad \begin{array}{l} a = -1/4 \\ b = -\sqrt{3}/4 \end{array}$$

donc $y_p = -\frac{1}{4}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$

Solution générale : $y = C_1 \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} e^{-2x} - \frac{1}{4}\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$

4.13 résoudre $y' = Ay$ avec $y(0) = y_0$

si $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ $\quad y_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

suite au no. 4.10, on sais que les valeurs propres sont $\lambda_{1,2} = \pm 2$ et que les vecteurs propres sonts

$$\vec{u} = t \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$$

donc $y = C_1 \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} e^{-2x}$ et $y_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

posons la condition initiale y_0

$$C_1 \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} e^0 + C_2 \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} e^0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} \sqrt{3} & 1 & 1 \\ 1 & -\sqrt{3} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\sqrt{3} & 0 \\ \sqrt{3} & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -\sqrt{3} & 0 \\ 0 & 4 & 1 \end{array}\right] L_2 - \sqrt{3} L_1 \sim \left[\begin{array}{cc|c} 1 & -\sqrt{3} & 0 \\ 0 & 1 & 1/4 \end{array}\right] \tfrac{1}{4} L_2$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & \sqrt{3}/4 \\ 0 & 1 & 1/4 \end{array}\right] L_1 + \sqrt{3} L_2 \qquad \begin{aligned} C_2 &= 1/4 \\ C_1 &= \sqrt{3}/4 \end{aligned}$$

Solution unique:

$$y = \frac{\sqrt{3}}{4} \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} e^{2x} + \frac{1}{4} \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} e^{-2x}$$

$$= \frac{1}{4} \begin{bmatrix} 3 \\ \sqrt{3} \end{bmatrix} e^{2x} + \frac{1}{4} \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}^{-2x}$$

5.1 $\sum_{n=1}^{+\infty} \frac{(-1)^n}{2n+1} x^n = f(x)$

$a_n = \frac{(-1)^n}{2n+1} \qquad a_{n+1} = \frac{(-1)^{n+1}}{2n+1+1}$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}}{2n+2} \cdot \frac{2n+1}{(-1)^n}\right| = \frac{(-1)2n+1}{2n+2}$

$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\frac{(-1)2n+1}{2n+2}\right| = \lim_{n\to+\infty}\frac{2n+1}{2n+2} \stackrel{R.H.}{=} \lim_{n\to+\infty}\frac{2}{2} = 1$

$\frac{1}{R} = 1 \qquad R=1 \qquad |x|<1.$

$\text{derivée} \Rightarrow f'(x) = \frac{(-1)^n}{2n+1} n x^{n-1} = \frac{(-1)^n n x^n}{x(2n+1)} \qquad a_n = \frac{n(-1)^n}{x(2n+1)}$

$\lim_{n\to+\infty}\left|\frac{f'(x)_{n+1}}{f'(x)_n}\right| = \lim_{n\to+\infty}\left|\frac{\cancel{(-1)^{n+1}}(n+1)}{x(2(n+1)+1)} \cdot \frac{x(2n+1)}{n\cancel{(-1)^n}}\right|$

$= \lim_{n\to+\infty}\left|\frac{(n+1)(2n+1)}{n(2n+3)}\right| = \lim_{n\to+\infty}\left|\frac{2n^2+3n+1}{2n^2+3n}\right| \stackrel{R.H.}{=} \lim_{n\to+\infty}\left|\frac{4n+3}{4n+3}\right|$

$= \lim_{n\to+\infty}|1| = 1 \qquad |x|<1.$

$R=1$

IL EST PLUS SIMPLE DE PRENDRE $a_n = \frac{1}{n^2+1}$

5.4 $\sum_{n=1}^{+\infty} \frac{1}{n^2+1}(x+1)^n = f(x)$

$a_n = \frac{(x+1)^n}{n^2+1} \quad a_{n+1} = \frac{(x+1)^{n+1}}{(n+1)^2+1}$

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty}\left|\frac{(x+1)^{n+1}}{(n+1)^2+1}\cdot\frac{n^2+1}{(x+1)^n}\right| = \lim_{n\to+\infty}\left|\frac{(x+1)(n^2+1)}{(n+1)^2+1}\right|$$

$$\stackrel{2\times R.H}{=} \lim_{n\to+\infty}\left|\frac{(x+1)\cdot 2}{2}\right| = \lim_{n\to+\infty}|(x+1)| = |x+1|$$

série converge si $|x+1|<1. \Rightarrow -1<x+1<1 \Rightarrow \boxed{-2<x<0}$ ↳ intervalle de convergence.

$R=1$ et c'est centré à -1.

dérivée $\Rightarrow f'(x) = \frac{n(x+1)^{n-1}}{n^2+1}$

$$\lim_{n\to+\infty}\left|\frac{f'(x)_{n+1}}{f'(x)_n}\right| = \lim_{n\to+\infty}\left|\frac{(n+1)(x+1)^n}{(n+1)^2+1}\cdot\frac{n^2+1}{n(x+1)^{n-1}}\right| = \lim_{n\to+\infty}\left|\frac{(x+1)^n}{(x+1)^{n-1}}\cdot\frac{n^3+n^2+n+1}{n^3+2n^2+2n}\right|$$

$$\stackrel{2\times R.H.}{=} \lim_{n\to+\infty}\left|(x+1)\cdot\frac{6n+2}{6n+2}\right| = \lim_{n\to+\infty}|(x+1)| = |x+1|$$

1ère dérivée converge si $|x+1|<1 \Rightarrow -1<x+1<1 \Rightarrow \boxed{-2<x<0.}$ ↳ intervalle de convergence

$R=1$ et c'est centré à -1

5.16 $y'' - 2(x-1)y' + 2y = 0$

$$y(x) = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$
$$y'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots$$
$$y''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots$$

① $2y(x) = 2a_0 + 2a_1x \overset{+2a_2x^2}{+} 2a_3x^3 + 2a_4x^4 + \dots$
② $(-2x+2)y'(x) = (-2x+2)a_1 + 2(-2x+2)a_2x + 3(-2x+2)a_3x^2 + 4(-2x+2)a_4x^3 + 5(-2x+2)a_5x^4 + \dots$
③ $y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + \dots$

② $= 2a_1 - 2xa_1 + 4a_2x - 4x^2a_2 + 6a_3x^2 - 6x^3a_3 + 8a_4x^3 - 8x^4a_4 + 10a_5x^4 - 10x^5a_5 + \dots$

①+②+③: $0 = Ly = (2a_0 + 2a_2 + 2a_1) + (2a_1 + 6a_3 - 2a_1 + 4a_2)x + (2a_2 + 12a_4 - 4a_2 + 6a_3)x^2 + (2a_3 + 20a_5 - 6a_3 + 8a_4)x^3 + (2a_4 + 30a_6 - 8a_4 + 10a_5)x^4 + \dots$

$Ly = (2a_0 + 2a_1 + 2a_2) + (4a_2 + 6a_3)x + (-2a_2 + 6a_3 + 12a_4)x^2 + (-4a_3 + 8a_4 + 20a_5)x^3 + (-6a_4 + 10a_5 + 30a_6)x^4 + \dots$

tous les coefficients de $x, x^2, x^3, \dots$ sont nuls.

$x^0 \Rightarrow 2a_0 + 2a_1 + 2a_2 = 0. \qquad a_2 = -a_0 - a_1$

$x^1 \Rightarrow 4a_2 + 6a_3 = 0. \qquad a_3 = \dfrac{-4a_2}{6} = \dfrac{-2}{3}a_2$

$x^2 \Rightarrow -2a_2 + 6a_3 + 12a_4 = 0. \qquad a_4 = \dfrac{6a_2}{12} = \dfrac{a_2}{2} = \dfrac{a_2}{6} - \dfrac{a_3}{2}$

$x^3 \Rightarrow -4a_3 + 8a_4 + 20a_5 = 0. \qquad a_5 = -\dfrac{a_2}{3}$

$x^s \Rightarrow 2a_s - 2sa_s + 2(s+1)a_{s+1} + (s+2)(s+1)a_{s+2} = 0.$

$$(s+2)(s+1)a_{s+2} = 2sa_s - 2a_s - 2(s+1)a_{s+1}$$

$$a_{s+2} = \frac{2(s-1)a_s - 2(s+1)a_{s+1}}{(s+2)(s+1)}$$

vérification

$(s=0)$.

$$a_2 = \frac{-2a_0 - 2a_1}{2} = -a_0 - a_1$$

$(s=2)$

$$a_4 = \frac{2a_2 - 6a_3}{12} = \frac{a_2}{6} - \frac{a_3}{2}$$

la solution est donc:

$$y(x) = a_0 + a_1 x + a_2 x^2 - \frac{2}{3} a_2 x^3 + \frac{a_2}{2} x^4 - \frac{a_2}{3} x^5 + \ldots$$

11.3

$$\begin{aligned} 2x_1 - x_2 + 5x_3 &= 4 \\ -6x_1 + 3x_2 - 9x_3 &= -6 \\ 4x_1 - 3x_2 \quad &= -2 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 5 \\ -6 & 3 & -9 \\ 4 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix}$$

puisque $|-6| > |2|$ et $|-6| > |4|$

$$P_1A = \begin{bmatrix} -6 & 3 & -9 \\ 2 & -1 & 5 \\ 4 & -3 & 0 \end{bmatrix}, \quad \text{où } P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_1P_1A = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -6 & 3 & -9 \\ 2 & -1 & 5 \\ 4 & -3 & 0 \end{bmatrix} = \begin{bmatrix} -6 & 3 & -9 \\ 0 & 0 & 2 \\ 0 & -1 & -6 \end{bmatrix}$$

$$P_2M_1P_1A = \begin{bmatrix} -6 & 3 & -9 \\ 0 & -1 & -6 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$A = P_1^{-1}M_1^{-1}P_2^{-1}U = LU$$

$$M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -2/3 & 0 & 1 \end{bmatrix} \qquad P^{-1} = P^T$$

$$L = P_1^TM_1^{-1}P_2^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -2/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 0 & 1 \\ -2/3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1/3 & 0 & 1 \\ 1 & 0 & 0 \\ -2/3 & 1 & 0 \end{bmatrix}$$

on resout le systéme $Ly = b$

$$\begin{bmatrix} -1/3 & 0 & 1 \\ 1 & 0 & 0 \\ -2/3 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix}$$

$y_2 = -6$
$y_3 = 2$
$y_1 = -6$

ensuite on résout $Ux = y$

$$\begin{bmatrix} -6 & 3 & -9 \\ 0 & -1 & -6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ -6 \\ 2 \end{bmatrix}$$

$x_3 = 1$
$x_2 = 0$
$x_1 = 1/2$

donc $x = \begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$

11.5
$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3{,}8 \\ 4x_1 + 3x_2 - x_3 &= -5{,}7 \\ 5x_1 + 10x_2 + 3x_3 &= 2{,}8 \end{aligned}$$

$$\begin{bmatrix} 1/2 & -1/2 & 1 \\ 1 & 3/4 & -1/4 \\ 1/2 & 1 & 3/10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1{,}9 \\ -1{,}425 \\ 0{,}28 \end{bmatrix}$$

pivot max $1^{ère}$ colonne après étalonnage est $a_{21} = 4$ (permute lignes 1-2)

$$P_1 A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & -1 & 2 \\ 5 & 10 & 3 \end{bmatrix} \quad \text{où} \quad P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_1 P_1 A = A_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/4 & 1 & 0 \\ -5/4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -1 \\ 1 & -1 & 2 \\ 5 & 10 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 & -1 \\ 0 & -7/4 & 9/4 \\ 0 & 25/4 & 17/4 \end{bmatrix}$$

Pour la sous-matrice $\begin{bmatrix} -7/4 & 9/4 \\ 25/4 & 17/4 \end{bmatrix}$ pivot max $1^{ère}$ colonne est $|25/4| = 6{,}25$ (permute lignes 2-3).

il faudrait étalonner les lignes 2 et 3

$$P_2 A_1 = \begin{bmatrix} 4 & 3 & -1 \\ 0 & 25/4 & 17/4 \\ 0 & -7/4 & 9/4 \end{bmatrix} \quad \text{où} \quad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_2 P_2 A_1 = U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7/25 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -1 \\ 0 & 25/4 & 17/4 \\ 0 & -7/4 & 9/4 \end{bmatrix} = \begin{bmatrix} 4 & 3 & -1 \\ 0 & 25/4 & 17/4 \\ 0 & 0 & 86/25 \end{bmatrix}$$

$M_2 P_2 M_1 P_1 A = U$ et $A = P_1^{-1} M_1^{-1} P_2^{-1} M_2^{-1} U = LU$.

$M_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 5/4 & 0 & 1 \end{bmatrix}$ $\quad M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -7/25 & 1 \end{bmatrix}$ $\quad P^{-1} = P^T$

donc, $L = P_1^T M_1^{-1} P_2^T M_2^{-1}$

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 5/4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7/25 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1/4 & 1 & 0 \\ 1 & 0 & 0 \\ 5/4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -7/25 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/4 & -7/25 & 1 \\ 1 & 0 & 0 \\ 5/4 & 1 & 0 \end{bmatrix}$$

$$Ly = b = \begin{bmatrix} 1/4 & -7/25 & 1 \\ 1 & 0 & 0 \\ 5/4 & 1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3{,}8 \\ -5{,}7 \\ 2{,}8 \end{bmatrix}$$

$\frac{y_1}{4} - \frac{7y_2}{25} + y_3 = 3{,}8$ $\quad y_1 = -5{,}7$ $\quad \frac{5y_1}{4} + y_2 = 2{,}8$

$y_3 = 8{,}004$ $\quad y_2 = 9{,}925$

$$Ux = y = \begin{bmatrix} 4 & 3 & -1 \\ 0 & 25/4 & 17/4 \\ 0 & 0 & 86/25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5{,}7 \\ 9{,}925 \\ 8{,}004 \end{bmatrix}$$

$4x_1 + 3x_2 - x_3 = -5{,}7$ $\quad \frac{25x_2}{4} + \frac{17x_3}{4} = 9{,}925$ $\quad \frac{86x_3}{25} = 8{,}004$

$x_1 = -0{,}848$ $\quad x_2 = 0{,}006$ $\quad x_3 = 2{,}327$

$$x = \begin{bmatrix} -0{,}848 \\ 0{,}006 \\ 2{,}327 \end{bmatrix}$$