D 4.1

MAT 2784 B

Devoirs #4 07.02.09 REMI VAILLANCOURT

3,11 $y_1(x) = 2$; $y_2(x) = \sin^2 x$, $y_3(x) = \cos^2 x$
fonction linéairement dépendante ou indépendante $\neq 0$

$y_1' = 0$ $y_2' = 2\sin x \cos x$ $y_3' = -2\cos x \sin x$
$y_1'' = 0$ $y_2'' = -2\cos x \sin x$ $y_3'' = 2\sin x \cos x$

$$W(y_1, y_2, y_3) = \begin{vmatrix} 2 & \sin^2 x & \cos^2 x \\ 0 & 2\sin x \cos x & -2\cos x \sin x \\ 0 & -2\cos x \sin x & 2\sin x \cos x \end{vmatrix}$$

$$l_3 + l_2 \Rightarrow \begin{vmatrix} 2 & \sin^2 x & \cos^2 x \\ 0 & 2\sin x \cos x & -2\cos x \sin x \\ 0 & 0 & 0 \end{vmatrix}$$

$$\det \begin{vmatrix} 2\sin x \cos x & -2\cos x \sin x \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$$

$W = 0$ donc les solutions sont dépendantes

vérification soit $A, b, c \in \mathbb{R}$

$$a\,y_1(x) + b\,y_2(x) + c\,y_3(x) = 0$$
$$2a + b\sin^2 x + c\cos^2 x = 0$$

Soit $a = 1$ $b = c = -2$
$$2 - 2(\sin^2 x + \cos^2 x) = 0$$

Comme $(a,b,c) \neq (0,0,0)$
les solutions sont dépendantes.

3,16 $W \Rightarrow$ linéairement indépendante

$x \;,\quad x\ln x \;,\quad x^2 \ln x \qquad e^{-2} < x < +\infty$

$$W(x) = \begin{vmatrix} x & x\ln x & x^2\ln x \\ 0 & \ln x + 1 & 2x\ln x + x \\ 0 & \frac{1}{x} & 2\ln x + 3 \end{vmatrix}$$

$$x \begin{vmatrix} 1 & \ln x & x\ln x \\ 1 & \ln x + 1 & 2x\ln x + x \\ 0 & \frac{1}{x} & 2\ln x + 3 \end{vmatrix} \qquad l_2 - l_1$$

$$x \begin{vmatrix} 1 & \ln x & x\ln x \\ 0 & 1 & x\ln x + x \\ 0 & 1/x & 2\ln x + 3 \end{vmatrix} \qquad x \cdot \det \begin{vmatrix} 1 & x\ln x + x \\ 1/x & 2\ln x + 3 \end{vmatrix}$$

$= (2\ln x + 3 - \ln x + 1)x$

$W(x) = x(\ln x + 2) \neq 0$

det $w = 0$ lorsque $x = 0$ et $x = e^{-2}$

$x = e^{-2}$ se situe $0 < x < \infty$

pour $x > e^{-2}$ $W \neq 0$ alors

les fonctions sont linéairement

indépendantes pour $e^{-2} < x < +\infty$

Alors $y_1 ; y_2 ; y_3 \Rightarrow C^n$ sur intervalle $\mathbb{R}$

où il y a une unique équation différentielle

$$-x(2 + \ln x)y''' + (3 + \ln x)y'' - \left(\frac{5}{x} + \frac{2\ln x}{x}\right)y' + \left(\frac{5}{x^2} + \frac{2\ln x}{x^2}\right)y = 0$$

3.18 $xy'' + y' = 0 \rightarrow y_1(x) = \ln x$

Vérifions: $x\left(\frac{-1}{x^2}\right) + \frac{1}{x} = 0$, $\frac{-1}{x} + \frac{1}{x} = 0$ ✓

On pose $y_2(x) = y_1(x) \cdot u(x)$

$$y_2' = y_1'u + y_1u'$$

$$xy_2'' = x(y_1''u + 2y_1'u' + y_1u'')$$

$$\underbrace{Ly_2}_{0} = u\underbrace{Ly_1}_{0} + y_1u' + 2xy_1'u' + xy_1u''$$

$$0 = (y_1 + 2xy_1')u' + (xy_1)u''$$

$$0 = (\ln x + 2x(\tfrac{1}{x}))u' + (x\ln x)u''$$

$$0 = (\ln x + 2)u' + (x\ln x)u''$$

$$v = u' \text{ et } v' = u''$$

$$\Rightarrow (\ln x + 2)v = -(x\ln x)\frac{dv}{dx}$$

$$\int \frac{(\ln x + 2)}{(x\ln x)}dx = -\int \frac{dv}{v}$$

$\rightarrow$ $u = \ln x$, $du = \frac{1}{x}dx$

$$\int \frac{u+2}{u}du$$

$$\int \frac{u}{u} + \frac{2}{u}\,du$$

$$= u + 2\ln u$$

$$-\ln v = \int \frac{2\,dx}{x\ln x} + \int \frac{dx}{x}$$

$$\ln v = -2\ln|\ln x| - \ln x$$

$$v = e^{-2\ln|\ln x| - \ln x}$$

$$= e^{-2\ln|\ln x|} \cdot e^{-\ln x}$$

$$v = (\ln x)^{-2} \cdot x^{-1}$$

$$\int v = \int \frac{1}{(\ln x)^2 \cdot x}$$

$u = \ln x$, $du = \frac{dx}{x}$

$$\int v = \int \frac{du}{u^2}$$

$$= \frac{-1}{\ln x} = u$$

$$y_2 = \ln x \cdot u$$

$$= \ln x \cdot \frac{-1}{\ln x}$$

$$= -1$$

✓

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3.24 $y'' - y' - 2y = 2xe^{-x} + x^2$

$x^2 - x - 2 = 0$ $\qquad y_h(x) = C_1 e^{-x} + C_2 e^{+2x}$

$(\lambda+1)(\lambda-2) = 0$

$\lambda_1 = -1 \quad \lambda_2 = +2$

Trouvons $y_p(x)$

$y_{p_1}(x)$ pour $2xe^{-x}$

$y_{p_1}(x) = (ax+b)e^{-x} = axe^{-x} + \underbrace{be^{-x}}$ est dans l'équation homogène donc on doit multiplier x

$\downarrow \cdot x$

$y_{p_1}(x) = x(ax+b)e^{-x} = ax^2e^{-x} + bxe^{-x}$ → pas dans y_h ok

$= (ax^2 + bx)e^{-x}$

$y'_{p_1}(x) = (2ax+b)e^{-x} - (ax^2+bx)e^{-x}$

$y''_{p_1}(x) = 2ae^{-x} - (2ax+b)e^{-x} - (2ax+b)e^{-x} + (ax^2+bx)e^{-x}$

$e^{-x}(2a - 2ax - b - 2ax - b + \cancel{ax^2} + \cancel{bx} - 2ax - b + \cancel{ax^2} + \cancel{bx} - \cancel{2ax^2} - \cancel{2bx}) = (0x^2 + 2x + 0)e^{-x}$

$(-6ax + 2a - 3b)\cancel{e^{-x}} = (0x^2 + 2x + 0b)\cancel{e^{-x}}$

$-6a\cancel{x} = 2\cancel{x}$ $\qquad 2a - 3b = 0$

$a = -\frac{1}{3}$ $\qquad +3b = +2\left(-\frac{1}{3}\right)$

$b = \frac{2}{9}$

$\boxed{\text{Alors } y_{p_1}(x) = \left(-\frac{1}{3}x^2 - \frac{2}{9}x\right)e^{-x}}$

$y_{p_2}(x)$ pour x^2

$y_{p_2}(x) = ax^2 + bx + c$ $\qquad$ Alors $y'' - y' - 2y = 1x^2 + 0x + 0c$

$y'_{p_2}(x) = 2ax + b$ $\qquad 2a - 2ax - b - 2ax^2 - 2bx - 2c = 1x^2 + 0x + 0c$

$y''_{p_2}(x) = 2a$ $\qquad 2a - b - 2c = 0c \quad -2ax - 2bx = 0x \quad -2ax^2 = 1x^2$

Alors, $a = -\frac{1}{2}$ $\cdot b = \frac{1}{2}$ $\cdot c = -\frac{3}{4}$

$\boxed{y_{p_2}(x) = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}}$

$y_p(x) = y_{p_1}(x) + y_{p_2}(x)$

$y_p(x) = \left(-\frac{1}{3}x^2 + \frac{2}{9}x\right)e^{-x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}$

la solution générale ⇒ $y(x) = C_1e^{-x} + C_2e^{+2x} + \left(-\frac{1}{3}x^2 - \frac{2}{9}x\right)e^{-x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}$

✓ 04.5

$\underline{3,26}$ $y'' + y = 2\cos x$ $y(0) = 1$ $y'(0) = 0$

tracer pour $y(x)$ $x \geq 0$

pour y_h

$y = e^{\lambda x}$ $y' = \lambda e^{\lambda x}$ $y'' = \lambda^2 e^{\lambda x}$

$y_h(x) = \lambda^2 e^{\lambda x} + e^{\lambda x} = 0$

$e^{\lambda x}(\lambda^2 + 1) = 0$

$\lambda_1 = i$ $\lambda_2 = -i$

$y_h(x) = C_1 \cos x + C_2 \sin x$

$r(x) = 2\cos x$

$r'(x) = -\sin x$

$r''(x) = -\cos x$

pour $y_p(x)$

$y_p(x) = ax\sin x + bx\cos x$

$y'_p(x) = a\sin x + ax\cos x + b\cos x - bx\sin x$

$y''_p(x) = a\cos x + a\cos x - ax\sin x - b\sin x - b\sin x - bx\cos x$

$y''_p + y_p = r(x)$

$2a\cos x \cancel{- ax\sin x} - 2b\sin x \cancel{- bx\cos x} \cancel{+ ax\sin x} \cancel{+ bx\cos x} = 2\cos x$

$2a\cos x - 2b\sin x = 2\cos x$

Alors $b = 0$ $a = 1$

$2\cos x = 2\cos x$

$y_p(x) = x\sin x$

$y_g(x) = C_1\cos x + C_2\sin x + x\sin x$

$y_g'(x) = -C_1\sin x + C_2\cos x + x\cos x$

$y(0) = C_1\cos(0) + C_2\sin(0) + 0\sin(0) = 1$

$\therefore C_1 = 1$

$y'(0) = -C_1\sin(0) + C_2\cos(0) + \cancel{0\cos(0)} = 0$

$C_2 = 0$

la solution unique

$y_g(x) = \cos x + x\sin x$ $\Rightarrow$

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3.20 $y'' + y = \frac{1}{\sin x}$

pour $y_h(x)$

$y = e^{\lambda x}$ $y' = \lambda e^{\lambda x}$ $y'' = \lambda^2 e^{\lambda x}$

$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0$

$\lambda^2 + 1 = 0$

$\lambda_{1,2} = \pm i \sqrt{-1}$

$y_h(x) = A\cos x + B\sin x$

pour $y_p(x)$

$y_p(x) = C_1 \cos x + C_2 \sin x$

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sin x \end{bmatrix}$$

$C_1'\cos x + C_2'\sin x = 0$ $\quad$ $C_2' = -C_1'\frac{\cos x}{\sin x}$

$-C_1'\sin x + C_2'\cos x = \frac{1}{\sin x}$

$-C_1'\sin x - C_1'\frac{\cos x}{\sin x} \cdot \cos x = \frac{1}{\sin x}$

$-C_1'\sin x - \frac{C_1'\cos^2 x}{\sin x} = \frac{1}{\sin x}$

$-C_1'\left(\sin x + \frac{\cos^2 x}{\sin x}\right) = \frac{1}{\sin x}$

$-C_1' = \frac{1}{\sin x}\left(\frac{1}{\sin x}\right) + \frac{1}{\sin x}\left(\frac{\sin x}{\cos^2 x}\right)$

$-C_1' = \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{1}{\sin^2 x + \cos^2 x} = 1$

$-C_1' = 1$

$C_1' = -1$ $\quad$ $C_2' = -(-1)\frac{\cos x}{\sin x}$

$C_2' = \frac{\cos x}{\sin x} = \operatorname{cotan} x$

$$C_1 = \int C_1' \, dx = -\int 1 \, dx = -x$$

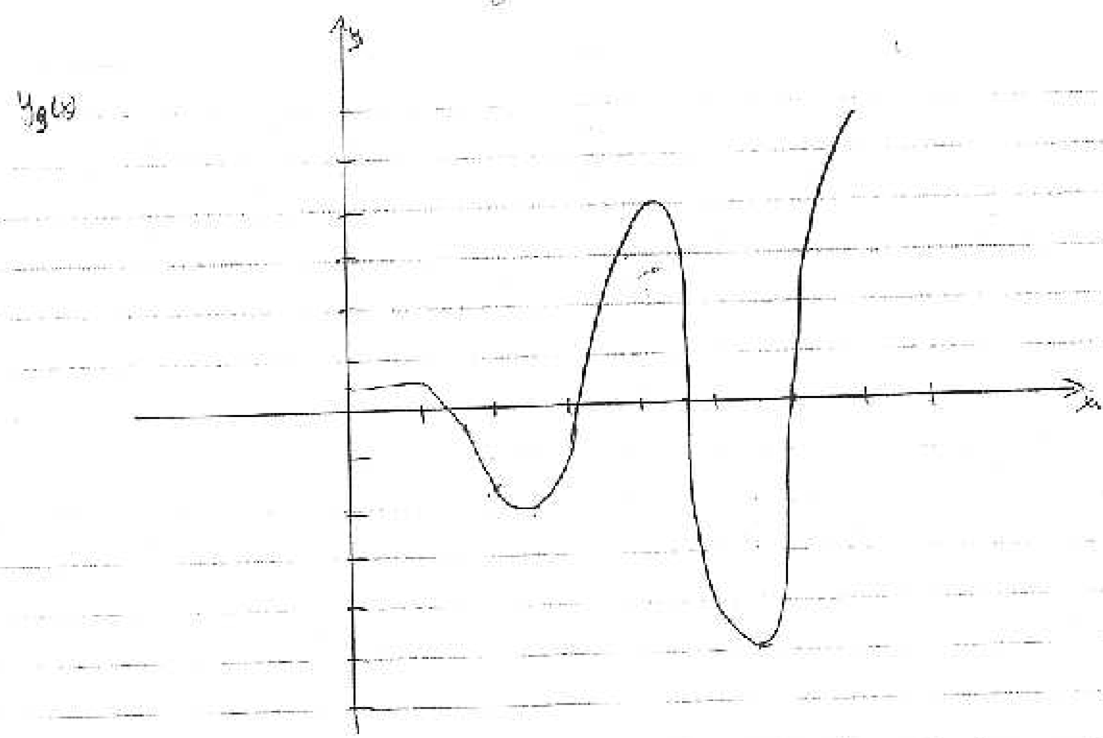
$$C_2 = \int C_2' \, dx = \int \cot x \, dx = \ln|\sin x|$$

Alors. $y_p = -x\cos x + \ln|\sin x| \sin x$

$y(x) = y_h + y_p$

$y(x) = A\cos x + B\sin x - x\cos x + \ln|\sin x| \cdot \sin x$

Graphique du n° 3.26



9.8 $f(8,1) = 16,94416$

$f(8,3) = 17,56492$

$f(8,6) = 18,50515$

$f(8,7) = 18,82091$

X_n	$f(X_n)$	1ère diff. dériv.	2e diff. dériv.	3e diff. dériv.
8,1	16,94416	3,1038	0,06	-0,00208
8,3	17,56492	3,1341	0,05875	
8,6	18,50515	3,1576		
8,7	18,82091			

$f(x_0, x_1) = 3,1038$

$f(x_1, x_2) = 3,1341$

$f(x_3, x_2) = 3,1576$

$f(x_0, x_1, x_2) = 0,06$

$f(x_1, x_2, x_3) = 0,05875$

$$P_3(x) = \overbrace{\underbrace{16,9440 + (x-8,1)(3,1041)}_{P_1(x)} + (x-8,1) \cdot 0,66\,(x-8,3)}^{P_2(x)} + (x-8,1)(x-8,3)(x-8,6)(-0,00208)$$

$x = 8,4$

$p(8,4) = 17,87\cancel{71}4248$

9.9 interpoler $(-1,2)$, $(0,0)$, $(1,-1)$, $(2,4)$ — D4.9 MAT 27841B

par un polynôme de Gregory-Newton de degré 3

i	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
0	-1	$\boxed{2}$	$\boxed{-2}$	$\boxed{1}$	$\boxed{5}$
1	0	0	-1	6	
2	1	-1	5		
3	2	4			

où $r = \frac{x - x_0}{h}$, $h = 1$ $\Rightarrow$ $r = (x+1)$

$$p_3(r) = 2 + r(-2) + \frac{r(r-1)(1)}{2} + \frac{r(r-1)(r-2)(5)}{6}$$

$$p_3(r) = 2 - 2r + \frac{r^2 - r}{2} + \frac{5}{6}(r^3 - 3r^2 + 2r)$$

9.8 $f(8,1) = 16,94410$ $f(8,3) = 17,56492$

$f(8,6) = 18,50515$ $f(8,7) = 18,82091$

$\Rightarrow$ interpoler $f(8,4)$ à l'aide des polynômes de Newton aux différences divisées

où $p_n(x) = f_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \dots$