SOLUTIONS

2007-01-31

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Devoir #3: Mat 2784 B

2.10 $y'' + 16y = 0$, $y(0) = 0$, $y'(0) = 1$

$y = e^{\lambda x}$

$e^{\lambda x}(\lambda^2 + 16) = 0$

$\lambda_{1,2} = \frac{-0 \pm \sqrt{0 - 4 \cdot 16}}{2} = \frac{\pm\sqrt{-64}}{2} = \pm \frac{8i}{2} = \pm 4i$ $\boxed{w = 4}$

$\therefore\ y(t) = c_1 \cos wt + c_2 \sin wt$

$y(t) = c_1 \cos 4t + c_2 \sin 4t$ $\boxed{p = \frac{2\pi}{w} = \frac{\pi}{2}}$

CI: $y(0) = c_1 + 0 = 0$

$c_1 = 0$

$y'(t) = -c_1 \sin(4t) \cdot 4 + c_2 \cos(4t) \cdot 4$

$y'(0) = 0 + c_2 \cdot 4 = 1$ $c_2 = 1/4$

Solution unique : $y(t) = \frac{1}{4} \sin 4t$

$A = \sqrt{c_1^2 + c_2^2} = 1/4$

$$\boxed{A = 1/4 \quad \text{et} \quad p = \pi/2}$$

2.12. Oscillateur, sous-amortissement critique
Trouver $T \geq 0$ telle que $|y(T)|$ soit maximum
Trouver le maximum, tracer solution $y(x)$ pour $x \geq 0$

$$y'' + 6y' + 9y = 0 \qquad y(0) = 0 \qquad y'(0) = 2$$

Posons $y = e^{\lambda x}$

$$e^{\lambda x}(\lambda^2 + 6\lambda + 9) = 0$$

$$(\lambda + 3)^2 = 0 \qquad \lambda_{1,2} = -3 = -\alpha$$

Solution générale: $y(t) = c_1 e^{-\alpha t} + c_1 t e^{-\alpha t}$

$$y(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$

Conditions initiales: $y(0) = c_1 + c_2(0) = 0$
$$c_1 = 0$$

$$y'(t) = c_1 e^{-3t} \cdot -3 + c_2 e^{-3t} + t \cdot c_2 \cdot (-3) e^{-3t} = 2$$

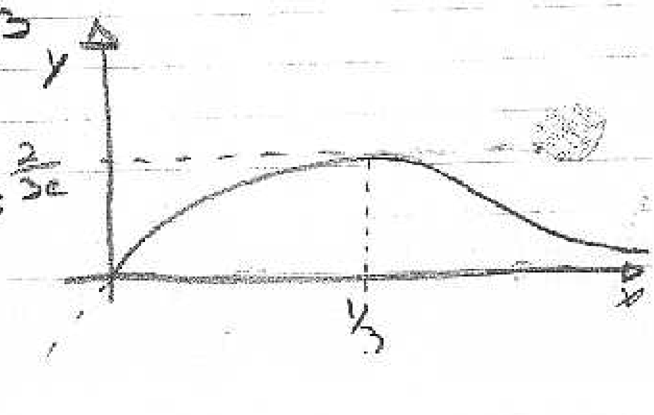
$$y'(0) = c_2 = 2$$

Solution unique: $y(t) = 2te^{-3t}$

max = ? $y'(t) = 2e^{-3t} - 6te^{-3t} = 0$
pente = 0
$$2\cancel{e^{-3t}} = 6t\cancel{e^{-3t}}$$
$$t = 1/3$$

Max = $y(1/3)$

$(y)\ y(1/3) = 2 \cdot \frac{1}{3} e^{-1} = \frac{2}{3e} = 0,245\,253$



2.13 $x^2 y'' + 3xy' - 3y = 0$

$y = x^m$ $\Rightarrow$ $x^m(m(m-1) + 3m - 3) = 0$

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

$m^2 + 2m - 3 = 0$

$(m-1)(m+3) = 0$

$m_1 = +1$

$m_2 = -3$

Solution générale : $y(x) = C_1 x^1 + C_2 x^{-3}$

2.20. $x^2 y'' + \frac{7}{2} x y' - \frac{3}{2} y = 0$; $y(4) = 1$, $y'(4) = 0$

Posons $y = x^m$

$y' = m x^{m-1}$

$y'' = m(m-1) x^{m-2}$

$\Rightarrow x^m\left(m(m-1) + \frac{7}{2}m - \frac{3}{2}\right) = 0$

$m^2 + \frac{5}{2}m - 3/2 = 0$

$2m^2 + 5m - 3 = 0$

$(2m-1)(m+3) = 0$

$m_1 = 1/2$

$m_2 = -3$

Solution générale: $y(x) = C_1 x^{1/2} + C_2 x^{-3}$

(1) $y(4) = C_1 \cdot 2 + \frac{C_2}{4^3} = 1$

$y'(x) = \frac{1}{2} \cdot C_1 x^{-1/2} + C_2 \cdot (-3) x^{-4}$

(2) $y'(4) = \frac{C_1}{4} + \frac{-3C_2}{4^4} = 0$

(2) $C_1 = \frac{3 C_2}{4^{43}} \cdot \cancel{4} = \frac{3C_2}{64}$

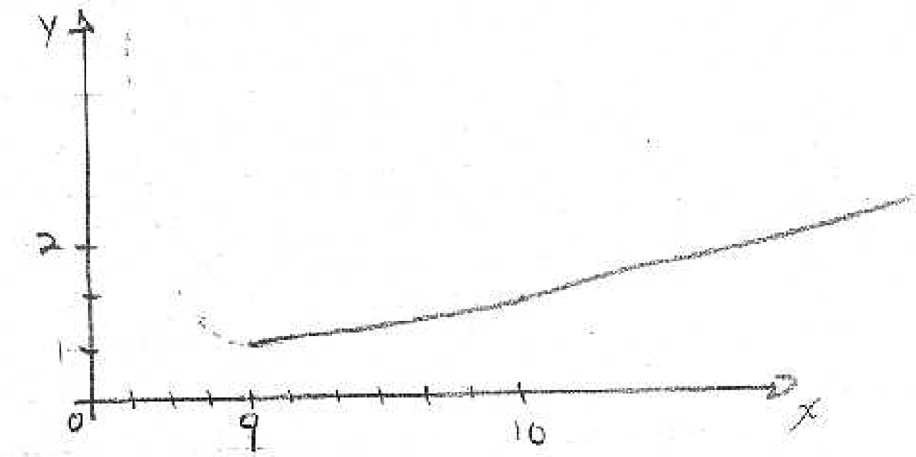
(1) $2 \cdot \left(\frac{3 C_2}{64}\right) + \frac{C_2}{64} = 1$

$C_2 = \frac{64}{7}$

$C_1 = \frac{3}{\cancel{64}} \cdot \frac{\cancel{64}}{7} = \frac{3}{7}$

solution: $y(x) = \frac{3}{7} x^{1/2} + \frac{64}{7} x^{-3}$
unique

graphe:



3.4 $y^{(4)} + y''' - 3y'' - y' + 2y = 0$

Posons $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{x\lambda}$, $y''' = \lambda^3 e^{\lambda x}$
$y^{(4)} = \lambda^4 e^{\lambda x}$

$$\cancel{e^{\lambda x}}\left(\lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2\right) = 0$$

$\lambda = 1 \Rightarrow 1 + 1 - 3 - 1 + 2 = 0$

$$\begin{array}{l|l}
\lambda^4 + \lambda^3 - 3\lambda^2 - \lambda + 2 & \lambda - 1 \\
-(\lambda^4 - \lambda^3) & \lambda^3 + 2\lambda^2 - \lambda - 2 \\
\quad 2\lambda^3 - 3\lambda^2 - \lambda + 2 & \\
\quad -(2\lambda^3 - 2\lambda^2) & \\
\qquad -\lambda^2 - \lambda + 2 & \\
\qquad -(-\lambda^2 + \lambda) & \\
\qquad\quad -2\lambda + 2 & \\
\qquad\quad -(-2\lambda + 2) & \\
\qquad\qquad 0 &
\end{array}$$

$$(\lambda - 1)\underbrace{(\lambda^3 + 2\lambda^2 - \lambda - 2)} = 0$$

si $\lambda = -1$ $\quad -1 + 2 + 1 - 2 = 0$

$$\begin{array}{l|l}
\lambda^3 + 2\lambda^2 - \lambda - 2 & \lambda + 1 \\
-(\lambda^3 + \lambda^2) & \lambda^2 + \lambda - 2 \\
\quad \lambda^2 - \lambda - 2 & \\
\quad -(\lambda^2 + \lambda) & \\
\qquad -2\lambda - 2 & \\
\qquad -(-2\lambda - 2) & \\
\qquad\quad 0 &
\end{array}$$

$$(\lambda - 1)(\lambda + 1)(\lambda^2 + \lambda - 2) = 0$$

$$(\lambda-1)^2(\lambda+1)(\lambda+2) = 0$$

$$\lambda_1 = \lambda_2 = 1 \qquad \lambda_3 = -1 \qquad \lambda_4 = -2$$

Solution générale : $y(t) = c_1 e^{x} + c_2 x e^{x} + c_3 e^{-x} + c_4 e^{-2x}$

#3.8 $y''' - 2y'' + 4y' - 8y = 0$

$y = e^{\lambda x}$

$$\lambda^3 - 2\lambda^2 + 4\lambda - 8 = 0$$

$$(\lambda-2)(\lambda^2+4) = 0$$

$\lambda_1 = 2 \qquad \lambda^2 + 4 = 0 \qquad \lambda = \sqrt{-4}$

$\lambda_{2,3} = \pm 2i$

Solution générale : $y(x) = c_1 e^{2x} + c_2 \cos(2x) + c_3 \sin(2x)$ (1)

$y'(x) = 2c_1 e^{2x} - c_2 \sin(2x)\cdot 2 + c_3 \cdot 2\cos(2x)$ (2)

$y''(x) = 4c_1 e^{2x} - 4c_2 \cos(2x) - 4\sin(2x)c_3$ (3)

$y(0) = 2, \quad y'(0) = 0, \quad y''(0) = 0$

(1) $c_1 + c_2 + c_3 \cdot 0 = 2$

$c_1 + c_2 = 2$

(2) $2c_1 - c_2 \cdot 0 + 2 \cdot c_3 = 0$

$2c_1 + 2c_3 = 0$

(3) $4c_1 - 4c_2 - 4 \cdot 0 \cdot c_3 = 0$

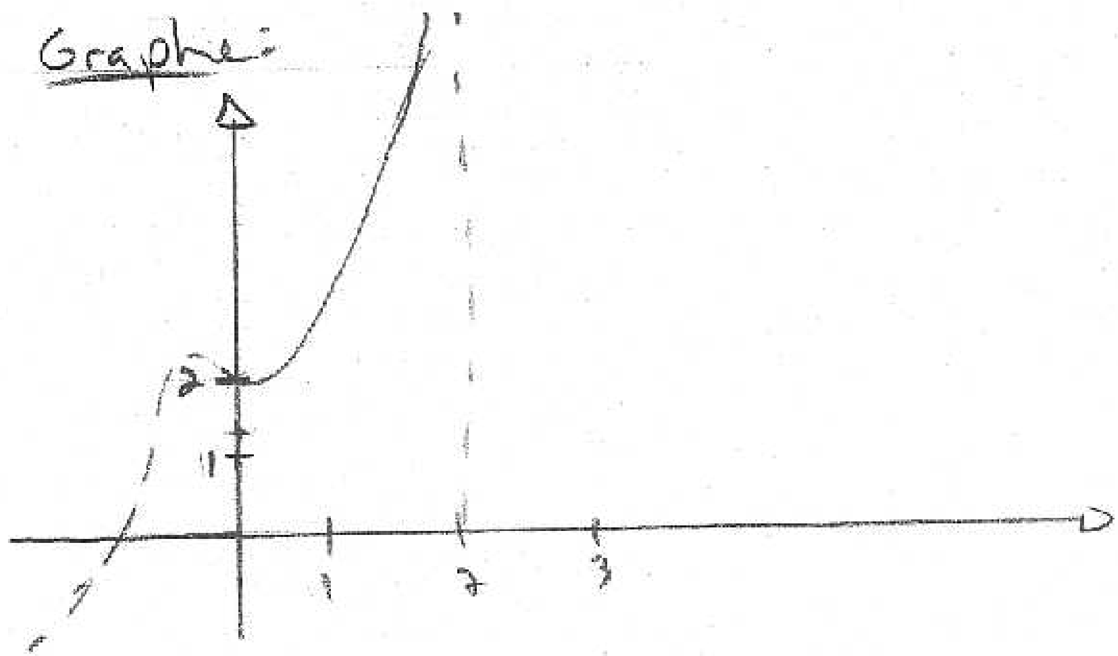
$c_1 = c_2$

$\Rightarrow \quad c_1 = c_2 = 1$

$\Rightarrow \quad c_3 = -1$

Solution unique : $y(x) = e^{2x} + \cos(2x) - \sin(2x)$

Graphe:



#9.1 $f(x) = \ln(x+1)$ $x_0 = 0$, $x_1 = 0{,}6$; $x_2 = 0{,}9$

degré 1: $p_1(x) = f(x_0) L_0(x) + f(x_1) L_1(x)$

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 0{,}6}{0 - 0{,}6} = \frac{x - 0{,}6}{0{,}6}$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x}{0{,}6}$$

$f(x_0) = 0$ $f(x_1) = \ln 1{,}6 = 0{,}47$

$$p_1(x) = \frac{0{,}47 \cdot x}{0{,}6} = 0{,}783\, x$$

degré 2: $p_2(x) = f(x_0)\,L_0(x) + f(x_1)\,L_1(x) + f(x_2)\,L_2(x)$

$f(x_0) = 0$
$f(x_1) = 0{,}47$
$f(x_2) = 0{,}6418$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-0{,}9)}{(0{,}6)(0{,}6-0{,}9)} = \frac{x(x-0{,}9)}{0{,}6\cdot(-0{,}3)}$$

$$= \frac{x^2-0{,}9x}{-0{,}18}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-0{,}6)}{(0{,}9-0)(0{,}9-0{,}6)} = \frac{x^2-0{,}6x}{0{,}27}$$

$$p_2(x) = \frac{0{,}47}{-0{,}18}\cdot(x^2-0{,}9x) + \frac{0{,}6418}{0{,}27}(x^2-0{,}6x)$$
$$= -0{,}234\,x^2 + 0{,}92378\,x$$

Évaluer en 0,45

$p_1(0{,}45) =$ $\qquad$ $p_2(0{,}45) =$