D1, 1

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MAT 2784 B
Devoir #1

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p. 289-290

1.3 $(1+x^2)y' = \cos^2 y$

$(1+x^2)\dfrac{dy}{dx} = \cos^2 y$

$\displaystyle\int \frac{dy}{\cos^2 y} = \int \frac{dx}{1+x^2}$

$\tan y = \arctan x + C.$

$y = \arctan(\arctan x + C).$

1.7 $y'\sin x - y\cos x = 0, \quad y(\pi/2) = 1$

$\dfrac{dy}{dx}\sin x = y\cos x$

$\displaystyle\int \frac{dy}{y} = \int \frac{\cos x}{\sin x}\,dx$

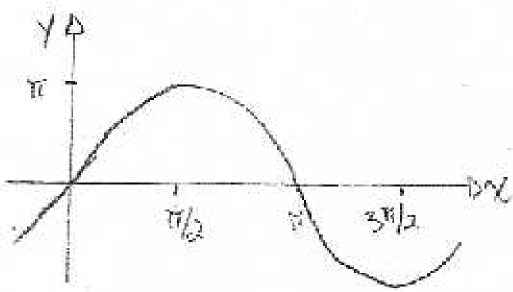
$\ln|y| = \int \cot x \, dx$

$\ln|y| = \ln|\sin x| + C$

$\ln(1) = \ln(\sin \pi/2) + C$

$C = 0$

$\Rightarrow \quad y = \sin x$



1.12 $xy' = y + x\cos^2(y/x)$

$$x\frac{dy}{dx} = y + x\cos^2(y/x)$$

$$xdy = [y + x\cos^2(y/x)]\,dx$$

$$xdy - [y + x\cos^2(y/x)]dx = 0$$

posons $y = xu$; $dy = xdu + udx$

$$x(xdu + udx) - \left[xu + x\cos^2\left(\frac{xu}{x}\right)\right]dx = 0$$

$$xdu + udx - [u + \cos^2 u]\,dx = 0$$

$$xdu = [u + \cos^2 u - u]\,dx$$

$$\int \frac{du}{\cos^2 u} = \int \frac{dx}{x}$$

$$\tan u = \ln|x| + C,$$

$$\tan(y/x) = \ln|x| + C$$

$$y = x\arctan[\ln|x| + C]$$

1.16 $(x^2+y^2)dx - 2xy\,dy = 0$, $y(1) = 2$

$$M\,dx - N\,dy = 0$$

$M = x^2 + y^2$ $\quad$ $N = -2xy$

$M_y = 2y$ $\quad$ $N_x = -2y$

$M_y \neq N_x$: pas exacte

$$\frac{M_y - N_x}{N} = \frac{2y - -2y}{-2xy} = \frac{4y}{-2xy} = \frac{-2}{x} = f(x)$$

facteur d'intégration: $\mu(x) = e^{\int -2/x\,dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$

$$\mu M dx + \mu N dy = 0$$

$$\left(\frac{x^2+y^2}{x^2}\right)dx - \frac{2xy}{x^2}\,dy = 0$$

$$\left(1+\frac{y^2}{x^2}\right)dx - \frac{2y}{x}\,dy = 0$$

$$u(x,y) = \int -\frac{2y}{x}\,dy + T(x)$$

$$= \frac{-y^2}{x} + T(x)$$

$$u_x = \frac{y^2}{x^2} + T'(x) = \cancel{N}\ M$$

$$= 1 + \frac{y^2}{x^2} \qquad \therefore\ T'(x) = 1 \Rightarrow T(x) = x$$

$$u(x,y) = \frac{-y^2}{x} + x$$

Solution générale: $\frac{-y^2}{x} + x = C$

Solution unique: $-\frac{(2)^2}{1} + 1 = C = -3$

$$\frac{-y^2}{x} + x = -3$$

1.17 $x(2x^2+y^2) + y(x^2+2y^2)y' = 0.$

$$x(2x^2+y^2) + y(x^2+2y^2)\frac{dy}{dx} = 0$$

$$x(2x^2+y^2)dx + y(x^2+2y^2)\,dy = 0$$

$$M\,dx + N\,dy = 0.$$

$M = 2x^3 + xy^2$ $\qquad$ $N = yx^2 + 2y^3$

$M_y = xy$ $\qquad$ $N_x = xy$

$M_y = N_x \therefore$ exacte.

$$u(x,y) = \int N(x,y)\,dy + T(x)$$
$$= \int (yx^2 + 2y^3)\,dy + T(x)$$
$$= \frac{y^2x^2}{2} + \frac{y^4}{2} + T(x)$$
$$u_x = y^2x + T'(x) = M$$
$$= 2x^3 + xy^2 \qquad \therefore\ T'(x) = 2x^3 \Rightarrow T(x) = \frac{x^4}{2}$$

$$u(x,y) = \frac{y^2x^2}{2} + \frac{y^4}{2} + \frac{x^4}{2}$$

Solution générale: $\frac{y^2x^2}{2} + \frac{y^4}{2} + \frac{x^4}{2} = C_1$

$y^2x^2 + y^4 + x^4 = C$ où $C = 2C_1$

1.24 $(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy = 0$, $y(0) = 2$

$M\,dx + N\,dy = 0.$

$M = 2x\cos y + 3x^2y$ $\qquad$ $N = x^3 - x^2\sin y - y$

$M_y = -2x\sin y + 3x^2$ $\qquad$ $N_x = 3x^2 - 2x\sin y$

$M_y = N_x \ \therefore$ exacte.

$$u(x,y) = \int M(x,y)\,dx + T(y)$$
$$= \int (2x\cos y + 3x^2y)\,dx + T(y)$$
$$= x^2\cos y + x^3y + T(y)$$
$$u_y = -x^2\sin y + x^3 + T'(y) = N$$
$$= x^3 - x^2\sin y - y \qquad \therefore\ T'(y) = -y \Rightarrow T(y) = -\frac{y^2}{2}$$

$$u(x,y) = x^2\cos y + x^3y - \frac{y^2}{2}$$

Solution générale: $x^2\cos y + x^3y - \frac{y^2}{2} = C$

solution unique: $0^2\cos 2 + 0^3 \cdot 2 - \frac{2^2}{2} = c = -2$.

$$x^2\cos y + x^3 y - \frac{y^2}{2} = -2$$

8.6 $f(x) = x^3 - x - 1 = 0$ sur $[1,2]$, $x_0 = 1$

$f(1) \cdot f(2) = -5 < 0$, donc par le corollaire 8.1, $f(x)$ admet un zéro entre 1 et 2

$x^3 = x + 1$

$x^2 = \frac{x+1}{x}$

$x = \sqrt{\frac{x+1}{x}}$

$g(x) = \left(1 + \frac{1}{x}\right)^{1/2}$ $\qquad g'(x) = \frac{1}{2}\left(1 + \frac{1}{x}\right)^{-1/2}(-x^{-2})$

$$g'(x) = \frac{-1}{2x^2} \cdot \frac{1}{\sqrt{1 + 1/x}}$$

Toutes les conditions du théorème du point fixe sont satisfaites, même la 3e :

$$|g'(x)| = \left| \frac{-1}{2x^2} \cdot \frac{1}{\sqrt{1+1/x}} \right| \leq K = 0,353553391 \ \forall x \in [1,2]$$

$x_{n+1} = g(x_n)$

$x_{n+1} = \left(1 + \frac{1}{x_n}\right)^{1/2}$

itérations n	racine x_n	écart $\lvert x_n - x_{n+1} \rvert$
0	1	—
1	1,414213562	0,414213562
2	1,306562965	0,107650597
3	1,32867109	0,022108125
4	1,323869976	0,004801114 $< 10^{-2}$
5	1,324900445	0,001030469

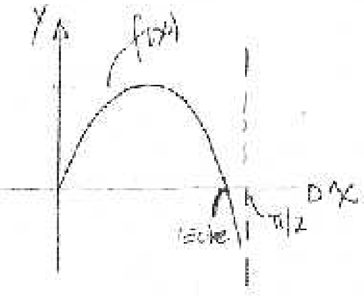
La solution à 10^{-2} près est 1,32.

8.10 $f(x) = 3x - \tan x$ 0 à 6 décimales Newton $x_0 = 1$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{où } f'(x_n) = 3 - \sec^2 x_n$$

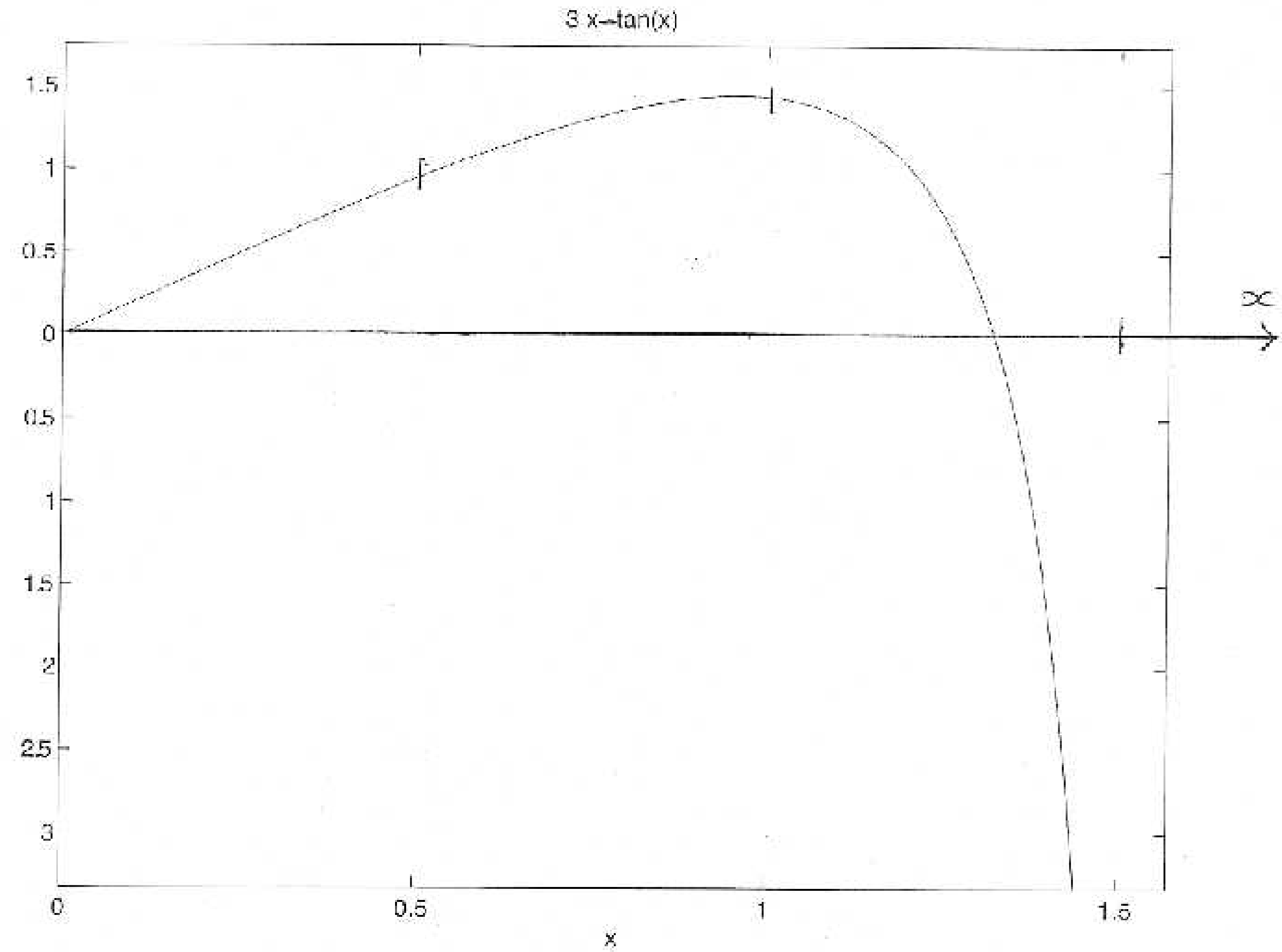
$$= x_n - \frac{3x_n - \tan x_n}{3 - \sec^2 x_n}$$

itérations n	racine x_n	écart $\lvert x_n - x_{n+1} \rvert$
0	1	
1	4,390196167	3,390196167
2	5,849288060	1,459091893
3	-4,239273627	10,08856169
4	-10,16577034	5,926433073
5	15,26716785	25,43290108
6	-10,766346622	26,03309241
7	-12,47781189	1,7199074
8	6,357548867	18,83056286
9	-3,167959106	9,504690943
10	1,572441774	4,731307532
11	1,574100204	0,000319736
12	1,577455308	0,003355304
13	1,584324845	0,006866537



Il est impossible de déterminer la racine avec la méthode de Newton.

$f(x) = 3x - \tan x$ admet une infinité de solutions !!



$x_0 = 0.5$

$x_1 = -0.0604863$

$x_2 = 0.0000741173$

$x_3 = -1.35718 \times 10^{-13}$

$x_4 = 0$

$x_0 = 1$ est un mauvais départ p.c.q. Newton saute d'une branche à l'autre de $\tan x$