Devoir #1

1.4 $(1+e^x)yy' = e^x$

$$y\frac{dy}{dx} = \frac{e^x}{1+e^x}$$

$$\int y\,dy = \int \frac{e^x}{1+e^x}\,dx$$

posons $u = 1+e^x$ $\quad \frac{du}{dx} = e^x \quad dx = \frac{du}{e^x}$

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{\cancel{e^x}}{u}\frac{du}{\cancel{e^x}} = \int \frac{du}{u}$$

$$= \ln|u| + C = \ln(1+e^x) + C$$

Donc,

$$\int y\,dy = \ln(1+e^x) + C$$

$$\frac{y^2}{2} = \ln(1+e^x) + C$$

$$y^2 = 2\ln(1+e^x) + C$$

$$y = \sqrt{2\ln(1+e^x) + C} \quad \checkmark$$

$$\boxed{y = \sqrt{\ln(1+e^x)^2 + C}}$$

$\underline{1.7}$ $y' \sin x - y \cos x = 0$, $y(\pi/2) = 1$

$$y' \sin x = y \cos x$$
$$\frac{dy}{dx} \sin x = y \cos x$$
$$dy \sin x = y \cos x \, dx$$

$$\int \frac{dy}{y} = \int \frac{\cos x}{\sin x} dx$$

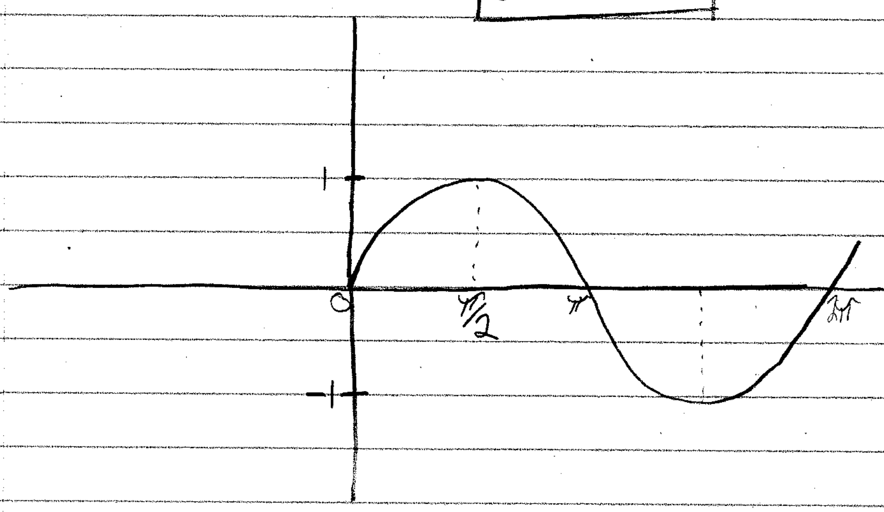
$$\ln|y| = \int \cot x \, dx$$

$$\ln|y| = \ln|\sin x| + C$$
$$e^{\ln|y|} = e^{\ln|\sin x| + C_1}$$

$$y = A \sin x \quad \text{où} \quad A = e^C$$

$$1 = A \sin\left(\frac{\pi}{2}\right)$$
$$1 = A(1)$$
$$A = 1$$

$$y = (1) \sin x \Rightarrow \boxed{y = \sin x}$$



✓

Hilroy

1.9 $(x^2-3y^2)dx + 2xy\,dy = 0$

$$M(x,y)dx + N(x,y)dy = 0$$

$$M_y = -6y \quad \neq \quad N_x = 2y$$

$$\frac{M_y - N_x}{N} = \frac{-6y-(2y)}{2xy} = \frac{-8y}{2xy} = \frac{-4}{x} = f(x)$$

$$\mu(x,y) = e^{\int -\frac{4}{x}dx} = e^{-4\int \frac{1}{x}dx} = e^{-4\ln x} = \cancel{e^{\ln}} x^{-4} = x^{-4}$$

$$\mu M dx + \mu N dy = x^{-4}(x^2-3y^2)dx + x^{-4}(2xy)dy = 0$$

ÉQ. EXACTE

$$u(x,y) = \int 2x^{-3}y\,dy + T(x)$$

$$= x^{-3}y^2 + T(x)$$

$$= -3x^{-4}y^2 + T'(x) = \mu M$$

$$\cancel{-3x^{-4}y^2} + T'(x) = x^{-2} - \cancel{3x^{-4}y^2}$$

$$T'(x) = x^{-2}$$

$$T(x) \;\; -x^{-1}$$

$$U(x,y) = x^{-3}y^2 - \frac{1}{x} = C$$

sol. générale

$$\boxed{\frac{y^2-x^2}{x^3} = C}$$ ✓

1.15 $yy' = -(x+2y)$, $y(1) = 1$

$$y\frac{dy}{dx} = -(x+2y)$$

$$y\,dy = -(x+2y)\,dx$$

$$y\,dy + (x+2y)\,dx = 0$$

$$(x+2y)\,dx + y\,dy = 0$$

$M(x,y) = (x+2y)$ $\quad N(x,y) = y$ $\quad$ (homogènes du degré 1)

Posons $y = ux$ et $dy = u\,dx + x\,du$

$$(x + 2(ux))\,dx + (ux)[u\,dx + x\,du] = 0$$

$$x(1+2u)\,dx + x(u^2dx + ux)\,du = 0$$

$$ux\,du + (u^2+2u+1)\,dx = 0$$

$$\frac{u}{u^2+2u+1}\,du + \frac{dx}{x} = 0$$

$$\int \frac{u}{(u+1)^2}\,du = -\int \frac{dx}{x} + C$$

$$\frac{1}{u+1} + \ln(u+1) = -\ln(x) - C$$

$$e^{\frac{1}{\frac{y}{x}+1}} + \ln\left(\frac{y}{x}+1\right) = -\ln(x) - C$$

verso $\longrightarrow$

$$e^{\frac{1}{y/x+1}}\left(\frac{y}{x}+1\right)=\frac{C}{x}$$

si $y(1)=1$

$x=1$

$y=1$

$$e^{\frac{1}{1+1}}\left(\frac{1}{1}+1\right)=\frac{C}{1}$$

$$e^{1/2}(2)=C$$

$$e^{\frac{1}{y/x+1}}\left(\frac{y}{x}+1\right)=\frac{2e^{1/2}}{x}$$

1.18 $(3x^2y^2-4xy)y' + 2xy^3 - 2y^2 = 0$

$$(2xy^3-2y^2)dx + (3x^2y^2-4xy)dy = 0$$

$$\overset{U_x}{M}dx + \overset{U_y}{N}dy = 0$$

$$M_x = 6xy^2 - 4y = N_y = 6xy^2 - 4y$$

$$u(x,y) = \int U_y\, dy + T(x)$$

$$= \int (3x^2y^2 - 4xy)\, dy + T(x)$$

$$= \frac{\cancel{3}x^2y^3}{\cancel{3}} - \frac{4xy^2}{\cancel{2}} + T(x) \Rightarrow x^2y^3 - 2xy^2 + T(x)$$

$$U_x(x,y) = 2xy^3 - 2y^2 + T'(x) = U_x$$

$$\cancel{2xy^3} - \cancel{2y^2} + T'(x) = 2xy^3 - 2y^2$$

$$T'(x) = 0$$

$$T(x) = 0$$

$$\boxed{U(x,y) = x^2y^3 - 2xy^2 = C}$$ ✓

EXACTE

1.20 $$\left(\frac{\sin 2x}{y} + x\right)dx + \left(y - \frac{\sin^2 x}{y^2}\right)dy = 0$$

$$U_x\,dx + U_y\,dy = 0$$

$$M dx + N dy = 0$$

$$M_y = -\frac{\sin 2x}{y^2} \qquad N_x = -\frac{2\sin(x)\cos(x)}{y^2}$$

$$M_y = -\frac{\sin(2x)}{y^2} = N_x = -\frac{\sin(2x)}{y^2}$$

$$u(x,y) = \int U_x\,dx + T(y)$$

$$= \int\left(\frac{\sin 2x}{y} + x\right)dx + T(y)$$

$$= \frac{\sin^2 x}{y} + \frac{1}{2}x^2 + T(y)$$

$$U_y(x,y) = \cancel{\frac{-\sin^2 x}{y^2}} + T'(y) = N = \cancel{\frac{-\sin^2 x}{y^2}} + y$$

$$T'(y) = y$$

$$T(y) = \frac{y^2}{2}$$

$$u(x,y) = \frac{\sin^2 x}{y} + \frac{x^2}{2} + \frac{y^2}{2}$$

$$\boxed{\frac{\sin^2 x}{y} + \frac{x^2+y^2}{2} = C}$$

Hi Joy

✓

7.5) $x_{n+1} = \sqrt{2x_n + 3}$ (récurrence de point fixe)

pour $f(x) = x^2 - 2x - 3 = 0$ converge sur l'intervalle $[2,4]$

Solⁿ: alors $g(x) = \sqrt{2x_n + 3}$

① $g(2) = \sqrt{2(2)+3} = \sqrt{7} \approx 2.64575$

$g(4) = \sqrt{2(4)+3} = \sqrt{11} \approx 3.316625$

alors $g(x)$ est croissante sur $[a,b]$: $2 \leq g(x) \leq 4$

② $g'(x) = \dfrac{1}{\sqrt{2x_n + 3}}$ alors $g'(x)$ existe sur $[2,4]$

③ $g'(2) = \dfrac{1}{\sqrt{2(2)+3}} = 0.37796 < 1$

$g'(4) = \dfrac{1}{\sqrt{2(4)+3}} = 0.30151 < 1$

$|g'(x)| < 1 \ \forall x \in [2,4]$

alors $g'(x)$ est décroissante $\rightarrow 0 < g'(x) \leq 0.3 < 1$

Donc $x_{n+1} = \sqrt{2x_n + 3}$ converge sur $[2,4]$ ✓

7.6) $f(x) = x^3 - x - 1 = 0$ sur $[1,2]$; $x_0 = 1$

Sol^n : $x^3 = x+1 \Rightarrow x^2 = \frac{x+1}{x} \Rightarrow x = \sqrt{\frac{x+1}{x}}$

$g(x) = \sqrt{\frac{x+1}{x}}$

① $g(1) = \sqrt{\frac{1+1}{1}} \rightarrow g(1) = \sqrt{2}$

$g(2) = \sqrt{\frac{3}{2}}$ alors $g(1)$ et $g(2)$ sont dans $[1,2]$

② $g(x) = \sqrt{\frac{x+1}{x}} \rightarrow g'(x) = \frac{1}{2}\left(\frac{x+1}{x}\right)^{-1/2} \cdot \frac{1}{x} + -\frac{(x+1)}{x^2}$

$g'(x) = \frac{-1}{2x^2} \cdot \sqrt{\frac{1}{\left(\frac{x+1}{x}\right)}}$

③ $g'(1) = \left| \frac{-1}{2} \cdot \sqrt{\frac{1}{2}} \right| = 0.35355339$

$g'(2) = \left| \frac{-1}{8} \cdot \sqrt{\frac{1}{3/2}} \right| = 0.10$

$\Big\} \; 0 < K < 1$

alors débutons les itérations :

$g(x_0) = g(1) = \sqrt{2/1} = \underline{1.4142136 = x_1}$

$g(x_1) = \sqrt{(1.4142136+1)/1.4142136} = \underline{1.30656295 = x_2}$

$g(x_2) = \sqrt{(1.3065.. + 1)/1.3065..} = \underline{1.32867109 = x_3}$

$g(x_3) = \sqrt{(1.32867.. + 1)/1.32867..} = \underline{1.32386998 = x_4}$

$g(x_4) = \sqrt{(1.32387.. + 1)/1.32387..} = \underline{1.32490044 = x_5}$

alors la réponse est de $\underline{1.32}$ ✓