Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et statistique
Faculty of Science
Mathematics and Statistics

Nom / Name : SOLUTIONS

No d'ét. / Stud. No.:

MAT 2784 B

Test mi-session 2
Durée: 80 min
Place: VNR 1075
23 mars 2011
17h30–18h50
Prof.: Rémi Vaillancourt

Midterm 2
Time: 80 min
Place: VNR 1075
23 March 2011
17:30–18:50

Instructions:

(a) *À livre fermé. Tout type de calculatrices permis.*
Closed book. Any type of calculators is allowed.

(b) *Répondre sur le questionnaire.*
Answer on the question sheets.

(c) *Les 6 questions sont d'égale valeur.*
The 6 questions have the same value.

(d) *Donner le détail de vos calculs.*
Show all computation.

(e) *Table à la fin.* / Table at the end.

(f) *Angles en RADIANS* / Angles in RADIAN measures.
Test: $\sin 1.123456789 = 0.90160112364453$

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
Total	/~~60~~ 50

585, av. King-Edward
Ottawa (Ontario) K1N 6N5 Canada
585 King Edward Avenue
Ottawa, Ontario K1N 6N5 Canada

(613) 562-5864 • Téléc./Fax (613) 562-5776
Courriel/Email: uomaths@science.uottawa.ca

Qu. 1. *Trouver h et n pour approcher l'intégrale à 10^{-4} près,*

Find h and n to approximate the integral to 4 decimals,

$$\int_1^2 x^2 \ln x \, dx,$$

par la méthode des trapèzes / by the composite trapezoidal rule :

$$\int_a^b f(x)\,dx = \frac{h}{2}\sum_{i=1}^{n}[f(x_{i-1})+f(x_i)] - \frac{(b-a)h^2}{12} f''(\xi), \quad a<\xi<b.$$

$f(x) = x^2 \ln x$

$f'(x) = 2x\ln(x) + x$

$f''(x) = 2\ln(x) + 3$

$\to f''(1) = 3$

$f''(2) = 2\ln(2)+3$

$M = \left|\max_{1\le x<2} f''(x)\right| = 2\ln(2)+3$

$\left|\frac{(b-a)h^2}{12} f''(\xi)\right| \le \frac{h^2}{12} M \le 10^{-4} \quad \to h = 0{,}016540237$

$\frac{1}{h} = 60{,}4586 \le n = 61$

Alors $\boxed{h = \frac{1}{61} \text{ et } n = 61}$

$h \simeq 0.01639$

Qu. 3. *Résoudre le problème aux valeurs initiales.* / Solve the initial value problem.

COEFF. INDÉT.

$$y'' - y' - 2y = e^x, \quad y(0) = 1, \quad y'(0) = 0.$$

$Ly = y'' - y' - 2y = 0 \quad \Rightarrow \quad \lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0 \quad \Rightarrow \lambda_1 = 2 \ , \ \lambda_2 = -1$

$y_h(x) = C_1 e^{2x} + C_2 e^{-x}$

$y_p(x) = a e^x$

$-2y_p = -2ae^x$

$-y_p' = -ae^x$

$y_p'' = ae^x$

$y_p'' - y_p' - 2y = ae^x - ae^x - 2ae^x = -2ae^x = e^x \quad \Rightarrow -2a = 1 \Rightarrow a = -1/2$

$y_p(x) = -\frac{1}{2} e^x$

$\Rightarrow y(x) = y_h(x) + y_p(x) = C_1 e^{2x} + C_2 e^{-x} - \frac{1}{2} e^x$ (solution générale)

$y(0) = C_1 + C_2 - \frac{1}{2} = 1 \quad \Rightarrow C_1 + C_2 = 3/2$

$y'(x) = 2C_1 e^{2x} - C_2 e^{x} - \frac{1}{2} e^x$

$y'(0) = 2C_1 - C_2 - \frac{1}{2} = 0 \quad \Rightarrow 2C_1 - C_2 = \frac{1}{2}$

$\begin{cases} C_1 + C_2 = 3/2 \\ 2C_1 - C_2 = 1/2 \end{cases}$ $\quad 3C_1 = 2 \Rightarrow C_1 = 2/3$

$C_2 = \frac{3}{2} - \frac{2}{3} = \frac{9-4}{6} = \frac{5}{6}$

$\therefore \ y(x) = \frac{2e^{2x}}{3} + \frac{5e^{-x}}{6} - \frac{e^x}{2}$ (solution unique)

Qu. 3. *Résoudre le problème aux valeurs initiales.* / Solve the initial value problem.

VARIATION DES PAR.

$$y'' - y' - 2y = e^x, \quad y(0) = 1, \quad y'(0) = 0.$$

Trouvons d'abord la solⁿ à l'équation homogène

$y'' - y' - 2y = 0$ On pose $y = e^{\lambda x}$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0 \quad \lambda_1 = 2 \quad \lambda_2 = -1$

$y_h(x) = c_1 e^{2x} + c_2 e^{-x}$

Par variation de paramètres

$y_p(x) = c_1(x) e^{2x} + c_2(x) e^{-x}$

$$\begin{bmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^x \end{bmatrix}$$

$$\widetilde{L_2 \to L_2 - 2L_1} \begin{bmatrix} e^{2x} & e^{-x} \\ 0 & -3e^{-x} \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^x \end{bmatrix}$$

$c_1' e^{2x} + c_2' e^{-x} = 0$

$-c_2' 3e^{-x} = e^x$

$c_2' = \frac{-e^x}{3e^{-x}} = \frac{-e^{2x}}{3}$

$c_1' e^{2x} - \frac{e^{-x} \cdot e^{2x}}{3} = 0$

$3c_1' e^{2x} - e^x = 0$

$c_1' = \frac{e^x}{3e^{2x}} = \frac{1}{3e^x} = \frac{1}{3} \cdot e^{-x}$

$c_1' = \frac{1}{3} \cdot e^{-x} \Rightarrow c_1 = -\frac{1}{3} e^{-x}$

$c_2' = \frac{-e^{2x}}{3} \Rightarrow c_2 = \frac{-e^{2x}}{6}$

$y_p(x) = -\frac{1}{3} e^{-x} e^{2x} - \frac{e^{2x}}{6} \cdot e^{-x} = -\frac{1}{3} e^x - \frac{e^x}{6} = e^x\left(-\frac{1}{3} - \frac{1}{6}\right) = -\frac{1}{2} e^x$

$y_g(x) = y_h(x) + y_p(x) = Ae^{2x} + Be^{-x} - \frac{1}{2} e^x$

$y_g'(x) = 2Ae^{2x} - Be^{-x} - \frac{1}{2} e^x$

$y(0) = 1 = A + B - \frac{1}{2} \Rightarrow B = 2A - \frac{1}{2} \qquad A + 2A - \frac{1}{2} = \frac{3}{2}$

$3A = 2 \quad A = \frac{2}{3}$

$y'(0) = 0 = 2A - B - \frac{1}{2} \qquad B = 2 \cdot \frac{2}{3} - \frac{1}{2} = \frac{5}{6}$

$$\boxed{y(x) = \frac{2}{3} e^{2x} + \frac{5}{6} e^{-x} - \frac{1}{2} e^x} \checkmark$$

Qu. 2. *Résoudre le problème aux valeurs initiales.* / Solve the initial value problem.

PAR LAPLACE

$$y'' - y' - 2y = e^x, \quad y(0) = 1, \quad y'(0) = 0.$$

$$s^2 Y(s) - s\,y(0) - y'(0) - [s\,Y(s) - y(0)] - 2\,Y(s) = \frac{1}{s-1}$$

$$(s^2 - s - 2)\,Y(s) = s - 1 + \frac{1}{s-1}$$

$$Y(s) = \frac{s-1}{(s+1)(s-2)} + \frac{1}{(s+1)(s-1)(s-2)}$$

$$\text{I}: \quad \frac{s-1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

2 dénominateurs distincts du 1^er^ degré $\Rightarrow$ ← MÉTHODE COURTE

$$s - 1 = (s-2)A + (s+1)B$$

$s = -1$: $-2 = -3A$ $\rightarrow A = 2/3$

$s = 2$: $1 = 3B$ $\rightarrow B = 1/3$

3 dén.

$$\text{II}: \quad \frac{1}{(s+1)(s-1)(s-2)} = \frac{C}{s+1} + \frac{D}{s-1} + \frac{E}{s-2}$$

$$1 = (s-1)(s-2)\,C + (s+1)(s-2)\,D + (s+1)(s-1)\,E$$

$s = -1$: $1 = (-2)(-3)\,C$ $\rightarrow C = 1/6$

$s = 1$: $1 = (2)(-1)\,D$ $\rightarrow D = -1/2$

$s = 2$: $1 = (3)(1)\,E$ $\rightarrow E = 1/3$

$$Y(s) = \frac{2}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2} + \frac{1}{6}\frac{1}{s+1} - \frac{1}{2}\frac{1}{s-1} + \frac{1}{3}\frac{1}{s-2}$$

$$= \frac{5}{6}\frac{1}{s+1} + \frac{2}{3}\frac{1}{s-2} - \frac{1}{2}\frac{1}{s-1}$$

$$y(t) = \frac{5}{6}e^{-t} + \frac{2}{3}e^{2t} - \frac{1}{2}e^{t}$$

Qu. 3. *Trouver la solution générale.* / Find the general solution.

$$y'' + y = \frac{1}{\cos x}.$$

par variation des paramètres

① Solution homogène λ de l'équation

$y'' + y = 0 \quad \lambda^2 + 1 = 0$

$\lambda_{1,2} = \pm i \; . \; \alpha = 0 \, , \beta = 1$

$y_{h(x)} = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$

$\boxed{y_h(x) = c_1 \cos x + c_2 \sin x}$ où $y_1(x) = \cos x$ et $y_2(x) = \sin x$

② sol^n particulière

$y_p(x) = c_1(x) \cos x + c_2(x) \sin x$

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\cos x} \end{bmatrix}$$

A orthogonale $\Rightarrow A^{-1} = A^T$

$$\begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\cos x} \end{bmatrix}$$

$\rightarrow c_1'(x) = -\frac{\sin x}{\cos x} = -\tan x \quad \boxed{c_1(x) = \ln|\cos x|}$

$\rightarrow c_2'(x) = \frac{\cos x}{\cos x} = 1 \quad \boxed{c_2(x) = x}$

$y_p(x) = \ln(\cos x) \cdot \cos x + x \sin x$

$y(x) = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x$

✓

Qu. 4. *Résoudre par Laplace.* / Solve by Laplace transform.

$$y(t) = \sin t + \int_0^t y(\tau)\sin(t-\tau)\,d\tau.$$

Note : $\mathcal{L}\{f * g\}(s) = F(s)G(s)$.

$y(t) = \sin t + y(t) * \sin t$

$\mathcal{L}(y(t)) = Y(s) = \frac{1}{s^2+1} + Y(s) * \frac{1}{s^2+1}$

$Y(s) - Y(s) * \frac{1}{s^2+1} = \frac{1}{s^2+1} \rightarrow Y(s)\left(1 - \frac{1}{s^2+1}\right) = \frac{1}{s^2+1}$

$Y(s) = \frac{\left(\frac{1}{s^2+1}\right)}{\left(1 - \frac{1}{s^2+1}\right)} = \frac{1}{s^2}$ $\boxed{Y(s) = \frac{1}{s^2}}$

$\mathcal{L}(Y(s)) = \boxed{y(t) = t}$ ✓

Qu. 5. *Résoudre par Laplace.* / Solve by Laplace transform.

$$y'' + 4y' = u(t-1), \quad y(0) = 0, \quad y'(0) = 0.$$

Note : $\mathcal{L}\{u(t-a)f(t-a)\}(s) = e^{-as}F(s)$.

$$s^2Y(s) - \cancel{sy(0)}^{\,0} - \cancel{y'(0)}^{\,0} + 4(sY(s) - \cancel{y(0)}^{\,0}) = \frac{e^{-s}}{s}$$

$$(s^2+4s)Y(s) = \frac{e^{-s}}{s} \rightarrow s(s+4)Y(s) = \frac{e^{-s}}{s}$$

$$Y(s) = e^{-s}\left(\frac{1}{s^2(s+4)}\right)$$

$$\rightarrow \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+4} = 1$$

$$As^2 + 4As + Bs + 4B + Cs^2 = 1$$

$$\rightarrow A + C = 0$$
$$4A + B = 0$$
$$4B = 1 \rightarrow \boxed{B = \frac{1}{4}}$$

$$4A + 1/4 = 0 \quad \boxed{A = -1/16}$$
$$-1/16 + C = 0 \quad \boxed{C = 1/16}$$

$$Y(s) = e^{-s}\left(\frac{-1}{16}\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{1}{s^2} + \frac{1}{16}\cdot\frac{1}{s+4}\right)$$

$$\mathcal{L}(Y(s)) = \boxed{y(t) = u(t-1)\left(-\frac{1}{16} + \frac{(t-1)}{4} + \frac{e^{-4(t-1)}}{16}\right)}$$

✓

Qu. 5. *Résoudre par Laplace.* / Solve by Laplace transform.

PROB. MODIFIÉ $$y'' + 4y = u(t-1), \quad y(0) = 0, \quad y'(0) = 0.$$

Note : $\mathcal{L}\{u(t-a)f(t-a)\}(s) = e^{-as}F(s)$.

$G(s) = y'' + 4y$

$= \mathcal{L}(y'') + 4\,\mathcal{L}(y)$

$= s^2\mathcal{L}(y) - sy(0) - y'(0) + 4\mathcal{L}(y)$

$G(s) = s^2 Y(s) \cancel{- sy(0) - y'(0)} + 4Y(s)$

$G(s) = s^2 Y(s) + 4Y(s)$

$G(s) = Y(s)(s^2+4)$

$Y(s) = \dfrac{G(s)}{(s^2+4)}$

$G(t) = u(t-1)$

$G(s) = \dfrac{e^{-s}}{s}$

$Y(s) = \dfrac{\left(\dfrac{e^{-s}}{s}\right)}{(s^2+4)} = \dfrac{e^{-s}}{(s)(s^2+4)}$

Fractions simples

$\dfrac{1}{(s)(s^2+4)} = \dfrac{A}{s} + \dfrac{Bs+C}{(s^2+4)} = \dfrac{A(s^2+4) + Bs^2 + Cs}{(s)(s^2+4)} = \dfrac{As^2 + 4A + Bs^2 + C}{(s)(s^2+4)}$

$\dfrac{(s^2)(A+B) + (s)(C) + (4A)}{(s)(s^2+4)}$

$$\left.\begin{aligned} A+B &= 0 \\ C &= 0 \\ 4A &= 1 \end{aligned}\right\} \begin{aligned} B &= -1/4 \\ C &= 0 \\ A &= 1/4 \end{aligned}$$

$Y(s) = e^{-s}\left(\dfrac{1}{4}\cdot\dfrac{1}{s} - \dfrac{1}{4}\dfrac{s}{(s^2+4)}\right)$

$$\boxed{Y(t) = u(t-1)\left(\frac{1}{4} - \frac{1}{4}\cos 2(t-1)\right)}$$

Qu. 6. *Soit le développement de Fourier-Legendre*
Consider the Fourier–Legendre expansion

$$\cos x = a_0\, P_0(x) + a_1\, P_1(x) + a_2\, P_2(x) + \dots, \quad -1 \le x \le 1$$

Calculer les coefficients de Fourier-Legendre a_0, a_1 *et* a_2.
Compute the Fourier–Legendre coefficients a_0, a_1 and a_2.

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1),$$

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} \cos x P_m(x)\, dx, \qquad m = 0, 1, 2, \dots$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} \cos x\,(1)\,dx = \frac{1}{2}\Big[\sin x\Big]_{-1}^{1} = \frac{2\sin(1)}{2} = \boxed{\sin(1)} = 0{,}841470$$

$$a_1 = \frac{3}{2}\int_{-1}^{1} \cos x\,(x)\,dx = \frac{3}{2}\Big[\cos x + x\sin x\Big]_{-1}^{1} = \frac{3}{2}[0] = \boxed{0}$$

$$a_2 = \frac{5}{2}\int_{-1}^{1} \cos x\left(\frac{1}{2}(3x^2-1)\right)dx = \frac{5}{4}\int_{-1}^{1} \cos x\,(3x^2-1)\,dx$$

$\int$ Par partie: $u = 3x^2 - 1 \quad dv = \cos x\, dx$
$du = 6x\,dx \quad v = \sin x$

$$= (3x^2-1)\sin x - \int 6x \sin x\, dx$$
$$= (3x^2-1)\sin x - \left(-6x\cos x - \int 6\cos x\, dx\right)$$
$$= (3x^2-1)\sin x + 6x\cos x - 6\sin x$$

$$a_2 = \frac{5}{4}\Big[6x\cos x + (3x^2-7)\sin x\Big]_{-1}^{1} = \frac{30\cos(1) - 20\sin(1)}{4}$$

$$-\left(\frac{-30\cos(1) + 50\sin(1)}{4}\right) = \boxed{15\cos(1) - 10\sin(1)}$$
$$= -0.310175$$

$$\boxed{\cos x = \sin(1) + \frac{(15\cos(1) - 10\sin(1))(3x^2-1)}{2}}$$

Note : $a_0 = 0$ p.c.q $x\cos x$ est impair