Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et statistique

Faculty of Science
Mathematics and Statistics

Nom / Name : SOLUTIONS

No d'ét. / Stud. No.: JAUNES

MAT 2784 B

Test mi-session 1
Durée: 80 min
Place: VNR 1075
16 février 2011
17h30–18h50
Prof.: Rémi Vaillancourt

Midterm 1
Time: 80 min
Place: VNR 1075
16 February 2011
17:30–18:50

Instructions:

(a) *À livre fermé. Tout type de calculatrices permis.*
Closed book. Any type of calculators is allowed.

(b) *Répondre sur le questionnaire.*
Answer on the question sheets.

(c) *Les 6 questions sont d'égale valeur.*
The 6 questions have the same value.

(d) *Donner le détail de vos calculs.*
Show all computation.

(e) *Table à la fin.* / Table at the end.

(f) *Angles en RADIANS* / Angles in RADIAN measures.
Test: $\sin 1.123456789 = 0.90160112364453$

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
Total	/60

L'équation différentielle homogène du 1er ordre, / The first-order homogeneous ODE,

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

admet le facteur d'intégration / admits the integrating factor

$$\mu(x) = e^{\int f(x)\,dx} \qquad \text{si/if} \qquad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x),$$

ou / or

$$\mu(y) = e^{-\int g(y)\,dy} \qquad \text{si/if} \qquad \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y).$$

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Qu. 1. (a) *Soit* / Consider $f(x) = x^3 - 5x^2 + 8x - 4.$

Calculer / Compute

$f(2) = \boxed{0}$

$f'(2) = \boxed{0}$ $f'(x) = 3x^2 - 10x + 8$

$f''(2) = \boxed{2}$ $f''(x) = 6x - 10$

(b) *Quelle est la multiplicité m du zéro $x = 2$ de $f(x)$?*
What is the multiplicity m of the zero $x = 2$ of $f(x)$?

$m = \boxed{2}$

(c) *Itérer deux fois à 6 décimales la méthode newtonienne modifiée avec m en (b) :*
Iterate twice to 6 decimals Newton's modified method with m in (b):

$$x_{n+1} = x_n - m\,\frac{f(x_n)}{f'(x_n)}, \qquad x_0 = 1.5.$$

$x_1 = \boxed{2.500\,000}$

$x_2 = \boxed{2.071430}$

(d) *Quelle est l'ordre de convergence p de la méthode en (c) ?*
What is the order of convergence p of the method in (c)?

$p = \boxed{2}$

Qu. 2. *Trouver un facteur d'intégration, rendre l'équation différentielle exacte et résoudre le problème à valeur initiale.*

Find an integration factor, make the differential equation exact and solve the initial value problem.

$$x^2y - x^3$$

$$(1 - x^2y)\,dx + x^2(y - x)\,dy = 0, \qquad y(1) = 1.$$

$M_y = -x^2$, $N_x = 2xy - 3x^2$

$$\frac{M_y - N_x}{N} = \frac{-x^2 - 2xy + 3x^2}{x^2(y-x)} = \frac{2x^2 - 2xy}{x^2(y-x)} = \frac{-2x\cancel{(-x+y)}}{x^2\cancel{(y-x)}} = \boxed{\frac{-2}{x}} = f(x)$$

$$\mu(x) = e^{-2\int \frac{1}{x}dx} = e^{-2\ln x} = \boxed{x^{-2}}$$

$$x^{-2}(1 - x^2y)dx + \cancel{x^2x^{-2}}(y-x)\,dy = 0$$

$$(x^{-2} - y)dx + (y-x)\,dy = 0$$

$$u(x,y) = \int (y-x)\,dy + T(x) = \frac{y^2}{2} - xy + T(x)$$

$$u_x(x,y) = -y + T'(x) = x^{-2} - y \qquad T'(x) = x^{-2}$$

$$T(x) = \int x^{-2}\,dx = -x^{-1} \cancel{+C}$$

Solution générale : $\boxed{\frac{y^2}{2} - xy - \frac{1}{x} = C}$

$$y(1) = 1 \Rightarrow \frac{(1)^2}{2} - (1)(1) - \frac{1}{(1)} = C$$

$$\frac{1}{2} - 1 - 1 = C \qquad C = -\frac{3}{2}$$

Solution unique : $\boxed{\frac{y^2}{2} - xy - \frac{1}{x} = -\frac{3}{2}}$

✓

Qu. 3. *Résoudre l'équation linéaire à valeur initiale.*

Solve the linear equation with given initial value.

$$xy' + 6y = 3x + 2, \qquad y(1) = 3.$$

$\Rightarrow y' + \frac{6}{x} y = 3 + \frac{2}{x}$ $\qquad f(x) = \frac{6}{x}$

$\mu(x) = e^{6\int \frac{1}{x} dx} = e^{6 \ln x} = \boxed{x^6}$

$x^6 y = \int x^6 (3 + 2x^{-1})\, dx$

$= \int 3x^6 + 2x^5\, dx = \frac{3}{7} x^7 + \frac{1}{3} x^6 + C$

$x^6 y = \frac{3}{7} x^7 + \frac{1}{3} x^6 + C$

Solution générale : $\boxed{y = \frac{3}{7} x + \frac{1}{3} + \frac{C}{x^6}}$

$y(1) = 3 \Rightarrow \qquad 3 = \frac{3}{7}(1) + \frac{1}{3} + \frac{C}{(1)^6}$

$3 = \frac{16}{21} + C \qquad \Rightarrow C = \frac{47}{21}$

Solution unique : $\boxed{y = \frac{3}{7} x + \frac{1}{3} + \frac{47}{21x^6}}$

Qu. 4. *Résoudre le problème aux valeurs initiales.*

Solve the initial value problem.

$$x^2y'' + 5xy' + 3y = 0, \quad x > 0, \quad y(1) = 2, \quad y'(1) = -1.$$

$\Rightarrow$ Equation d'Euler-Cauchy $(y = x^m)$

$$m_{1,2} = \frac{1-a}{2} \pm \frac{\sqrt{(a-1)^2 - 4b}}{2}$$

$$= \frac{1-5}{2} \pm \frac{\sqrt{(5-1)^2 - 4 \cdot 3}}{2} = \frac{-4}{2} \pm \frac{\sqrt{4}}{2} = -2 \pm 1 \qquad m_1 = -1, \quad m_2 = -3$$

2 Solutions indépendantes : $y_1(x) = x^{-1}$, $y_2(x) = x^{-3}$

Solution générale : $\boxed{y(x) = c_1 x^{-1} + c_2 x^{-3}}$

$y(1) = 2 \Rightarrow \quad 2 = c_1(1)^{-1} + c_2(1)^{-3}$

$\boxed{2 = c_1 + c_2} \quad \Rightarrow c_1 = 2 - c_2$

$y'(x) = -c_1 x^{-2} - 3c_2 x^{-4}$

$y'(1) = -1 \quad \Rightarrow \quad -1 = -c_1(1)^{-2} - 3c_2(1)^{-4}$

$\boxed{-1 = -c_1 - 3c_2} \quad \Rightarrow -1 = -(2 - c_2) - 3c_2$

$-1 = -2 + c_2 - 3c_2$

$1 = -2c_2$

$\boxed{c_2 = \frac{-1}{2}}$ & $\boxed{c_1 = \frac{5}{2}}$

Solution unique : $\boxed{y(x) = \frac{5}{2x} - \frac{1}{2x^3}}$

Qu. 5. *Résoudre le problème aux valeurs initiales.*

Solve the initial value problem.

$$y'' - 2y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 1.$$

$\cancel{e^{\lambda x}}(\lambda^2 - 2\lambda + 2) = 0 \Rightarrow \dfrac{-a}{2} \pm \dfrac{\sqrt{a^2 - 4b}}{2}$

$\dfrac{-(-2)}{2} \pm \dfrac{\sqrt{(-2)^2 - 4\cdot 2}}{2} = 1 \pm \dfrac{\sqrt{-4}}{2}$

$= 1 \pm i$

2 solutions complexes : $y_1(x) = e^x \cos x$, $y_2(x) = e^x \sin x$

Solution générale : $\boxed{y(x) = c_1 e^x \cos x + c_2 e^x \sin x}$

$y(0) = 2 \Rightarrow 2 = c_1 e^0 \cos(0) + \cancel{c_2 e^0 \sin(0)}^{\,0}$

$\boxed{2 = c_1}$

$y'(x) = 2e^x \cos x - 2e^x \sin x + c_2 e^x \cos x + c_2 e^x \sin x$

$y'(0) = 1 \Rightarrow 1 = 2e^0 \cos(0) - \cancel{2e^0 \sin(0)}^{\,0} + c_2 e^0 \cos(0) + \cancel{c_2 e^0 \sin(0)}^{\,0}$

$1 = 2\cdot 1 + c_2 \cdot 1$

$\boxed{-1 = c_2}$

Solution unique : $\boxed{y(x) = 2e^x \cos x - e^x \sin x}$

Qu. 6. (a) *Itérer 4 fois la récurrence de point fixe à 5 décimales près.*

Iterate 4 times the fixed point recurrence. Use at least 5 decimals.

$$x_{n+1} = g(x_n), \quad x_0 = 1, \quad \text{avec / with} \quad g(x) = \frac{5}{x^2} + 2.$$

et calculer l'erreur si : / and compute the error if: $p = g(p) = 2.690647.$

$x_1 = \boxed{7,00000\,00}$ $\quad x_1 - p = \boxed{4,309353}$

$x_2 = \boxed{2,10204\,08}$ $\quad x_2 - p = \boxed{-0,588606}$

$x_3 = \boxed{3,13158\,63}$ $\quad x_3 - p = \boxed{0,440939}$

$x_4 = \boxed{2,50984\,86}$ $\quad x_4 - p = \boxed{-0,180798}$

$x_5 = \boxed{2.793733}$ $\quad x_5 - p = \boxed{0.103087}$

(b) *Calculer* / Compute

$g'(x_5) = \boxed{-0,458611} \approx g'(p) = -0.513369.$ $\qquad g'(x) = \frac{-10}{x^3}$

(c) *Quel est l'ordre p de convergence de la méthode ?*
What is the order p of convergence of the method?

$|g'(p)| = \left|\frac{-10}{(p)^3}\right| = 0,513369 \neq 0$ $\rightarrow$ L'ordre de convergence est $\boxed{1}$.

Integration

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \tan x\,dx = -\ln|\cos x| + c$$

$$\int \cot x\,dx = \ln|\sin x| + c$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + c$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + c$$

$$\int \sin^2 x\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + c$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + c$$

$$\int \tan^2 x\,dx = \tan x - x + c$$

$$\int \cot^2 x\,dx = -\cot x - x + c$$

$$\int \ln x\,dx = x\ln x - x + c$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$$

Laplace Transform: General Formulas

Formula	Name, Comments
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity
$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$	s-Shifting (First Shifting Theorem)
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$	Differentiation of Function
$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Integration of Function
$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$	t-Shifting (Second Shifting Theorem)
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\,d\tilde{s}$	Differentiation of Transform Integration of Transform
$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$ $= \int_0^t f(t-\tau)g(\tau)\,d\tau$ $\mathcal{L}(f*g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution
$\mathcal{L}(f) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	f Periodic with Period p

$$\sin x \sin y = \tfrac{1}{2}[-\cos(x+y) + \cos(x-y)]$$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\sin x \cos y = \tfrac{1}{2}[\sin(x+y) + \sin(x-y)]$$