Université d'Ottawa • University of Ottawa
Faculté des sciences Mathématiques et statistique
Faculty of Science Mathematics and Statistics

Place / Seat

Nom/Name : SOLUTIONS

No d'ét./Stud. No.:

Examen final
Durée: 3h
Place: MNT 203
2011.04.27, 19h–22h
Prof.: Rémi Vaillancourt

MAT 2784 B

Final Exam
Time: 3h
Place: MNT 203
2011.04.27, 7–10 PM

Instructions:

(a) *À livre fermé. Tout type de calculatrices permis.*
Closed book. Any type of calculators is allowed.

(b) *Répondre sur le questionnaire.*
Answer on the question sheets.

(c) *Les 8 questions sont d'égale valeur.*
The 8 questions have the same value.

(d) *Donner le détail de vos calculs.*
Show all computation.

(e) *Tables* / Tables : p. 12–13.

(f) *Angles en RADIANS* / Angles in RADIAN measures.
Test: $\sin 1.123456789 = 0.90160112364453$

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
Total	/80

Vecteurs propres multiples et vecteurs propres généralisés

Soit une matrice A qui admet une valeur propre λ de multiplicité r et un *seul* vecteur propre associé $\boldsymbol{u}_1$. On construit les vecteurs propres généralisés $\{\boldsymbol{u}_2, \dots, \boldsymbol{u}_r\}$ solutions des systèmes

$$(A-\lambda I)\boldsymbol{u}_2 = \boldsymbol{u}_1, \quad (A-\lambda I)\boldsymbol{u}_3 = \boldsymbol{u}_2, \quad \dots, \quad (A-\lambda I)\boldsymbol{u}_r = \boldsymbol{u}_{r-1}.$$

Le vecteur propre $\boldsymbol{u}_1$ et les vecteurs propres généralisés engendrent r solutions de $\boldsymbol{y}' = A\boldsymbol{y}$ linéairement indépendantes :

$$\boldsymbol{y}_1(x) = e^{\lambda x}\boldsymbol{u}_1, \quad \boldsymbol{y}_2(x) = e^{\lambda x}(x\boldsymbol{u}_1 + \boldsymbol{u}_2), \quad \boldsymbol{y}_3(x) = e^{\lambda x}\left(\frac{x^2}{2}\boldsymbol{u}_1 + x\boldsymbol{u}_2 + \boldsymbol{u}_3\right), \quad \dots.$$

585, av. King-Edward
Ottawa (Ontario) K1N 6N5 Canada
585 King Edward Avenue
Ottawa, Ontario K1N 6N5 Canada
(613) 562-5864 • Téléc./Fax (613) 562-5776
Courriel/Email: uomaths@science.uottawa.ca

Qu. 1. *Résoudre le problème à valeur initiale.* / Solve the initial value problem.

$$\underbrace{(2x^2y - 2y + 5)}_{M(x,y)}\,dx + \underbrace{(2x^3 + 2x)}_{N(x,y)}\,dy = 0, \qquad y(1) = 0.$$

$M(x,y)\,dx + N(x,y)\,dy = 0$

$M_y = 2x^2 - 2$ & $N_x = 6x^2 + 2$ $\quad M_y \neq N_x \Rightarrow$ Pas exactes

$$\frac{M_y - N_x}{N} = \frac{2x^2 - 2 - 6x^2 - 2}{2x^3 + 2x} = \frac{-4x^2 - 4}{2x(x^2+1)} = \frac{-4\cancel{(x^2+1)}}{2x\cancel{(x^2+1)}} = \frac{-2}{x} = f(x)$$

$$\mu = e^{\int f(x)dx} = e^{-2\int \frac{1}{x}dx} = \boxed{x^{-2}}$$

$\mu M(x,y)\,dx + \mu N(x,y)\,dy = 0$

$\Rightarrow x^{-2}(2x^2y - 2y + 5)dx + x^{-2}(2x^3 + 2x)dy = 0$

$(2y - 2x^{-2}y + 5x^{-2})dx + (2x + 2x^{-1})dy = 0$

$u(x,y) = \int (2x + 2x^{-1})\,dy + T(x) = 2xy + 2x^{-1}y + T(x)$

$u_x(x,y) = 2y - 2x^{-2}y + T'(x) = 2y - 2x^{-2}y + 5x^{-2} \quad \Rightarrow T'(x) = 5x^{-2}$

$T(x) = \int 5x^{-2}\,dx = -\frac{5}{x} + C$

$u(x,y) = 2xy + \frac{2y}{x} - \frac{5}{x} = C \quad \Leftarrow$ solution générale

$y(1) = 0$: $2(1)(0) + \frac{2(0)}{(1)} - \frac{5}{(1)} = C \quad \Rightarrow \boxed{C = -5}$

Solution unique : $\boxed{2xy + \frac{2y}{x} - \frac{5}{x} = -5}$

Qu. 2. *Résoudre le problème à valeur initiale.* / Solve the initial value problem.

$$x^2 y'' + 4\,x y' - 2\,y = 0, \quad y(1) = 1, \quad y'(1) = 2. \qquad y = x^m$$

$$m_{1,2} = \frac{1-4}{2} \pm \frac{\sqrt{(4-1)^2 - 4\cdot(-2)}}{2} = -\frac{3}{2} \pm \frac{\sqrt{17}}{2} \qquad \lambda_1 = -\frac{3}{2} + \frac{\sqrt{17}}{2}$$

$$\lambda_2 = -\frac{3}{2} - \frac{\sqrt{17}}{2}$$

Solution générale : $y(x) = C_1 x^{-\frac{3}{2}+\frac{\sqrt{17}}{2}} + C_2 x^{-\frac{3}{2}-\frac{\sqrt{17}}{2}}$

$y(1) = 1 = C_1 + C_2$ $\qquad y'(x) = \left(-\frac{3}{2} + \frac{\sqrt{17}}{2}\right) C_1 x^{-\frac{5}{2}+\frac{\sqrt{17}}{2}} + \left(-\frac{3}{2} - \frac{\sqrt{17}}{2}\right) C_2 x^{-\frac{5}{2}-\frac{\sqrt{17}}{2}}$

$y'(1) = 2 = \left(-\frac{3}{2} + \frac{\sqrt{17}}{2}\right) C_1 + \left(-\frac{3}{2} - \frac{\sqrt{17}}{2}\right) C_2$ $\qquad \Rightarrow C_1 = \frac{1}{2} + \frac{7\sqrt{17}}{34}$

$$C_2 = \frac{1}{2} - \frac{7\sqrt{17}}{34}$$

Solution unique : $\boxed{y(x) = \left(\frac{1}{2} + \frac{7\sqrt{17}}{34}\right) x^{-\frac{3}{2}+\frac{\sqrt{17}}{2}} + \left(\frac{1}{2} - \frac{7\sqrt{17}}{34}\right) x^{-\frac{3}{2}-\frac{\sqrt{17}}{2}}}$

$$y(x) \simeq 1.3489\, x^{0,5616} - 0.3489\, x^{-3,5616}$$

Qu. 3. *Trouver la solution générale.* / Find the general solution.

$$y'' + 4y' + 4y = \frac{e^{-2x}}{x^2}.$$

$\cancel{e^{\lambda x}}(\lambda^2 + 4\lambda + 4) = 0 \Rightarrow (\lambda+2)^2 = 0 \quad \lambda_1 = \lambda_2 = -2$

Solution / de l'équation ~~homogène~~ homogène : $y_h(x) = c_1 e^{-2x} + c_2 x e^{-2x}$

Solution particulière : $y_p(x) = C_1(x) e^{-2x} + C_2(x) x e^{-2x}$

$$\begin{bmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-2x} x^{-2} \end{bmatrix}$$

$$\cancel{e^{-2x}} \begin{bmatrix} 1 & x \\ -2 & 1-2x \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \cancel{e^{-2x}} \begin{bmatrix} 0 \\ x^{-2} \end{bmatrix} \Rightarrow C_1' + C_2' x = 0 \Rightarrow C_1' = -C_2' x$$

$$-2C_1' + C_2' - 2xC_2' = x^{-2}$$

$$\cancel{-2(-C_2' x)} + C_2' \cancel{- 2C_2' x} = x^{-2}$$

Alors $C_2' = x^{-2}$ & $C_1' = -x^{-2} x = -\frac{1}{x}$

$C_1(x) = \int -\frac{1}{x}\,dx = -\ln|x|$ & $C_2(x) = \int \frac{1}{x^2}\,dx = \frac{-1}{x}$

$\Rightarrow y_p(x) = -\ln|x|\, e^{-2x} - e^{-2x} = -e^{-2x}(\ln|x| + 1)$

Solution générale : $y(x) = y_h(x) + y_p(x)$

$$\boxed{y(x) = c_1 e^{-2x} + c_2 x e^{-2x} - e^{-2x}(\ln|x| + 1)}$$

$$= c_3 e^{-2x} + c_2 x e^{-2x} - e^{-2x} \ln|x|,$$

où $c_3 = c_1 - 1$

Qu. 4. *Résoudre* / Solve :

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

n.b. : *Voir page titre* / See front page.

$$\det(A-\lambda I) = \begin{vmatrix} (2-\lambda) & 1 & 0 \\ 0 & (2-\lambda) & 1 \\ 0 & 0 & (2-\lambda) \end{vmatrix} = (2-\lambda)(2-\lambda)(2-\lambda)$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 2$$

$(A-\lambda I)\vec{u} = \vec{0}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $u_2 = 0$, $u_3 = 0$, Posons $u_1 = 1$ $\quad \vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$(A-\lambda I)\vec{v} = \vec{u}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = 1$, $v_3 = 0$, Posons $v_1 = 0$ $\quad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$(A-\lambda I)\vec{w} = \vec{v}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $v_2 = 0$, $v_3 = 1$, Posons $v_1 = 0$ $\quad \vec{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{Y(x) = c_1 e^{2x}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2x}\left(x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + c_3 e^{2x}\left(\frac{x^2}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)}$$ ← Solution générale

Solution unique : $x = 0$

$1 = c_1$

$-2 = c_2$

$3 = c_3$

$$\boxed{Y(x) = 1e^{2x}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2e^{2x}\left(x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + 3e^{2x}\left(\frac{x^2}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)}$$ ← solution unique

$((t-1)+1)((t-1)+1)$

$1-(t-1)^2 \cancel{+}2(t-1) \cancel{+} 1$

$2-(t-1)-1$

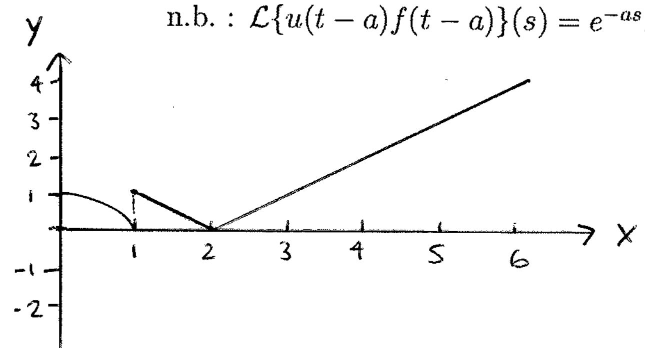
$u(t-a)(t-a)^n = n!e^{-as}/s^{(n+1)}$

Qu. 5. *Tracer $h(t)$ sur $[0,6]$ (2/10), développer $h(t)$ suivant la fonction $u(t-a)$ d'Heaviside (3/10) et trouver la transformée de Laplace de $h(t)$ (5/10).*

Sketch the graph of $h(t)$ on $[0,6]$ (2/10), expand $h(t)$ in terms of Heaviside's function $u(t-a)$ (3/10) and find the Laplace transform of $h(t)$ (5/10).

$$h(t) = \begin{cases} 1-t^2, & 0 \le t < 1, \\ 2-t, & 1 \le t < 2, \\ t-2, & 2 \le t. \end{cases}$$

n.b. : $\mathcal{L}\{u(t-a)f(t-a)\}(s) = e^{-as}F(s)$.



$h(t) = 1-t^2 - u(t-1)(1-t^2) + u(t-1)(2-t) - u(t-2)(2-t) + u(t-2)(t-2)$

$= 1-t^2 - u(t-1)(1-((t-1)+1)^2) + u(t-1)(2-((t-1)+1)) - u(t-2)(-1)(t-2) + u(t-2)(t-2)$

$= 1-t^2 - u(t-1)(-(t-1)^2 - 2(t-1)) + u(t-1)(1-(t-1)) + 2u(t-2)(t-2)$

$= 1-t^2 + u(t-1)(t-1)^2 + 2u(t-1)(t-1) + u(t-1) - u(t-1)(t-1) + 2u(t-2)(t-2)$

$h(t) = 1-t^2 + u(t-1)(t-1)^2 + u(t-1)(t-1) + u(t-1) + 2u(t-2)(t-2)$

$$\boxed{h(s) = \frac{1}{s} - \frac{2}{s^3} + \frac{2e^{-s}}{s^3} + \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} + \frac{2e^{-2s}}{s^2}}$$

Formules

$t^n = \dfrac{n!}{s^{(n+1)}}$

$u(t-a)(t-a)^n = \dfrac{n!e^{-as}}{s^{(n+1)}}$

Qu. 6. *Résoudre par transformation de Laplace.* / Solve by Laplace transforms.

$$y'' + 5y' + 6y = u(t-1), \qquad y(0) = 0, \quad y'(0) = 1.$$

n.b. : $\mathcal{L}\{u(t-a)\}(s) = \frac{1}{s} e^{-as}$.

$$\mathcal{L}(y(t)) := s^2 Y(s) - \cancel{s y(0)}^{0} - y'(0) + 5(sY(s) - \cancel{y(0)}^{0}) + 6Y(s) = \frac{e^{-s}}{s}$$

$$(s^2+5s+6)Y(s) - 1 = \frac{e^{-s}}{s} \Rightarrow (s+2)(s+3)Y(s) - 1 + e^{-s}\left(\frac{1}{s}\right)$$

$$Y(s) = \frac{1}{(s+2)(s+3)} ① + e^{-s}\left(\frac{1}{s(s+2)(s+3)} ②\right)$$

$$Y(s)^{①} = \frac{A}{s+2} + \frac{B}{s+3} = 1$$

$$As + 3A + Bs + 2B = 1$$

$$s^1: A + B = 0$$

$$s^0: 3A + 2B = 1$$

$$\boxed{A = 1 \quad B = -1}$$

$$Y(s)^{②} = \frac{C}{s} + \frac{D}{s+2} + \frac{E}{s+3} = 1$$

$$Cs^2 + 5Cs + 6C + Ds^2 + 3Ds + Es^2 + 2Es = 1$$

$$s^2: C + D + E = 0$$

$$s^1: 5C + 3D + 2E = 0$$

$$s^0: 6C = 1 \quad \Rightarrow C = 1/6$$

$$5C + 3D + 2E = 0$$

$$-2C - 2D - 2E = 0$$

$$3C + D = 0 \quad \Rightarrow D = -3(1/6) = -1/2 \quad \& \quad E = -1/6 + 1/2 = 1/3.$$

$$Y(s) = \frac{1}{s+2} - \frac{1}{s+3} + e^{-s}\left(\frac{1}{6}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s+3}\right)$$

$$\mathcal{L}^{-1}(Y(s)) = y(t) = e^{-2t} - e^{-3t} + u(t-1)\left(\frac{1}{6} - \frac{e^{-2(t-1)}}{2} + \frac{e^{-3(t-1)}}{3}\right)$$

✓

Qu. 7. *Soit* / Let $f(x) = x^2 \ln x$.

Par Richardson, extrapoler $f'(2)$ *obtenue par différence centrée avec* $h = 0.2, 0.1, 0.05$.
By Richardson, extrapolate $f'(2)$ obtained by central difference with $h = 0.2, 0.1, 0.05$.

$$N_1(0.2) = N(0.2) = \frac{1}{0.4}[f(2.2) - f(1.8)] = \boxed{4.779262074700506}$$

$$N_1(0.1) = N(0.1) = \frac{1}{0.2}[f(2.1) - f(1.9)] = \boxed{4.774255805871046}$$

$$N_1(0.05) = N(0.05) = \frac{1}{0.1}[f(2.05) - f(1.95)] = \boxed{4.773005414952762}$$

Ensuite / Next

$$N_2(0.2) = N_1(0.1) + \frac{N_1(0.1) - N_1(0.2)}{3} = \boxed{4.772587049594560}$$

$$N_2(0.1) = N_1(0.05) + \frac{N_1(0.05) - N_1(0.1)}{3} = \boxed{4.772588617980001}$$

Enfin / Finally

$$N_3(0.2) = N_2(0.1) + \frac{N_2(0.1) - N_2(0.2)}{15} = \boxed{4.772588722539030}$$

Calculer $f'(2)$ *exactement et vérifier que* $N_3(0.2)$ *est exacte à 6 décimales près :*
Compute $f'(2)$ exactly and verify that $N_3(0.2)$ is exact to 6 decimals:

$$A := \frac{d}{dx}(x^2 \ln x)\Big|_{x=2} = \boxed{4.772588722239782}$$

$$\left|A - N_3(0.2)\right| = \boxed{-2.992486258790450\text{e-}10} \leq 10^{-9}.$$

Qu. 8. MATLAB **ode23** *pour* / for $y' = f(x,y)$, $y(x_0) = y_0$:

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + (1/2)h, y_n + (1/2)k_1), \\ k_3 &= hf(x_n + (3/4)h, y_n + (3/4)k_2), \\ k_4 &= hf(x_n + h, y_n + (2/9)k_1 + (1/3)k_2 + (4/9)k_3), \end{aligned}$$

$$y_{n+1} = y_n + (2/9)k_1 + (1/3)k_2 + (4/9)k_3,$$

avec estimation de l'erreur locale / with local error estimate:

$$E_{n+1} = -\frac{5}{72}k_1 + \frac{1}{12}k_2 + \frac{1}{9}k_3 - \frac{1}{8}k_4.$$

Compléter les 5 petites cases pour l'équadif et les 8 ***grandes cases à 6 décimales****:*

Fill the 5 small boxes for the ode and the 8 **long boxes to six decimals**:

$$y' = y^3 - y - 2x, \qquad y(1) = 1, \qquad h = 0.1.$$

Pour / For $n = 0$:

$f(x,y) = \boxed{y^3 - y - 2x}$ $x_0 = \boxed{1.0}$ $y_0 = \boxed{1.0}$ $h = \boxed{0.10}$

$k_1 = \boxed{-0.200000000000000}$

$k_2 = \boxed{-0.227100000000000}$

$k_3 = \boxed{-0.240855941452808}$

$k_4 = \boxed{-0.251126180175707}$

$y_1 = \boxed{0.772808470465419}$

$E_1 = \boxed{-4.071098616819369\text{e-}04}$

Pour / For $n = 1$:

$x_1 = \boxed{1.10}$ $y_1 = \boxed{0.772808470465419}$

$k_1 = \boxed{-0.251126180175707}$