le 16 mars 2011

MAT2784B - DEVOIR #7.

$\boxed{5.57}$ $y(t) = \cos 3t + 2\int_0^t y(\tau)\cos 3(t-\tau)\,d\tau$

$$Y(s) = \frac{s}{s^2+9} + 2Y(s)\frac{s}{s^2+9}$$

$$Y(s) - 2Y(s)\frac{s}{s^2+9} = \frac{s}{s^2+9}$$

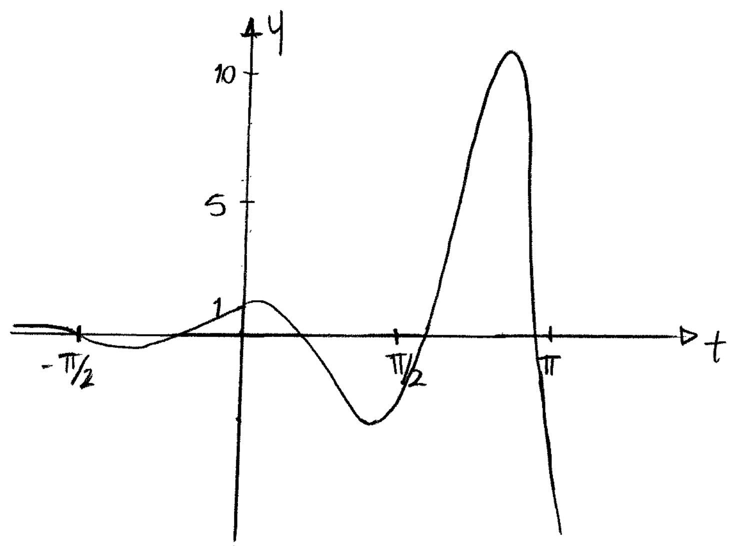
$$Y(s)\left(1 - 2\frac{s}{s^2+9}\right) = \frac{s}{s^2+9}$$

$$Y(s)\,\frac{s^2+9-2s}{s^2+9} = \frac{s}{s^2+9} \Rightarrow Y(s) = \frac{s\cancel{(s^2+9)}}{\cancel{(s^2+9)}(s^2-2s+9)}$$

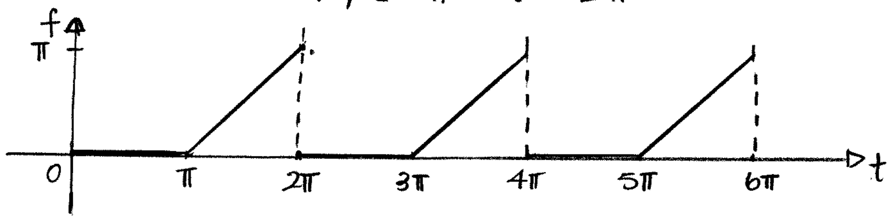
$$Y(s) = \frac{s+1-1}{s^2-2s+9-1+1}$$

$$\Rightarrow Y(s) = \frac{s-1+1}{(s-1)^2+8} = \frac{s-1}{(s-1)^2+8} + \frac{1}{(s-1)^2+8}\left(\frac{\sqrt{8}}{\sqrt{8}}\right) \quad \mathcal{L}\ F(s-1) = e^t f(t)$$

$$\mathcal{L}(Y(s)) = \boxed{y(t) = e^t\cos(\sqrt{8}t) + \frac{1}{\sqrt{8}}e^t\sin(\sqrt{8}t)}$$



$\boxed{5.64}$ $f(t) = \begin{cases} 0, & \text{si } 0 < t < \pi \\ t-\pi, & \text{si } \pi < t < 2\pi \end{cases}$



$p = 2\pi$

$$\mathcal{L}(f)(s) = \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} \cancel{e^{-st}(0)}^{\,0}\, dt + \int_{\pi}^{2\pi} e^{-st}(t-\pi)\, dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\int_{\pi}^{2\pi} e^{-st} t\, dt - \int_{\pi}^{2\pi} e^{-st}\pi\, dt\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[-\frac{e^{-st}t}{s} - \frac{e^{-st}}{s^2}\Bigg|_{\pi}^{2\pi} + \frac{\pi e^{-st}}{s}\Bigg|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\left(\frac{-e^{-2\pi s}2\pi}{s} - \frac{e^{-2\pi s}}{s^2}\right) - \left(\frac{-e^{-\pi s}\pi}{s} - \frac{e^{-\pi s}}{s^2}\right) + \left(\frac{\pi e^{-2\pi s}}{s}\right) - \left(\frac{\pi e^{-\pi s}}{s}\right)\right]$$

$$= \frac{1}{1-\cancel{e^{-2\pi s}}}\left(\cancel{e^{-2\pi s}}\left(\frac{-\pi}{s} - \frac{1}{s^2}\right) + \frac{e^{-\pi s}}{s^2}\right)$$

$$\boxed{\mathcal{L}(f)(s) = \frac{1}{e^{-2\pi s}-1}\left(\frac{-\pi}{s} - \frac{1}{s^2} + \frac{e^{\pi s}}{s^2}\right)}$$

$\boxed{6.2}$ $\sum_{n=1}^{\infty} \frac{2^n}{n3^{n+3}} x^n = f(x)$

$a_n = \frac{2^n}{n3^{n+3}}$ $\quad a_{n+1} = \frac{2^{n+1}}{(n+1)3^{n+1+3}} = \frac{2^{n+1}}{(n+1)3^{n+4}}$

$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}}{(n+1)3^{n+4}} \cdot \frac{n3^{n+3}}{2^n}\right| = \frac{2\ n}{3\ (n+1)}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2n}{3(n+1)}\right| \overset{R.H.}{=} \lim_{n\to\infty} \frac{2}{3} = \boxed{\frac{2}{3}} = \frac{1}{R} \Rightarrow \boxed{R = \frac{3}{2}}$ ✓

dérivée : $f'(x) = \frac{2^n}{n3^{n+3}} nx^{n-1} = \frac{2^n nx^n}{xn3^{n+3}} \Rightarrow a_n = \frac{2^n n}{xn3^{n+3}}$

$a_{n+1} = \frac{2^{n+1}(n+1)}{x(n+1)3^{n+4}}$

$\lim_{n\to\infty}\left|\frac{f'(x)_{n+1}}{f'(x)_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}\cancel{(n+1)}}{\cancel{x}\cancel{(n+1)}3^{n+4}} \cdot \frac{\cancel{x}\cancel{n}3^{n+3}}{2^n\cancel{n}}\right| = \boxed{\frac{2}{3}} = \frac{1}{R'} \Rightarrow |x| < \frac{3}{2}$

$\Rightarrow \boxed{R = \frac{3}{2}}$ ✓

la 1ère dérivée converge sur $-\frac{3}{2} < x < \frac{3}{2}$ (Rayon $\frac{3}{2}$ centrée à $x=0$)

$\boxed{6.5}$ $\sum_{n=3}^{\infty} \frac{n(n-1)(n-2)}{4^n} x^n = f(x)$

$a_n = \frac{n(n-1)(n-2)}{4^n}$ $\quad a_{n+1} = \frac{(n+1)n(n-1)}{4^{n+1}}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\cancel{n}\cancel{(n-1)}}{4^{n+1}} \cdot \frac{4^n}{\cancel{n}\cancel{(n-1)}(n-2)}\right|$

$$\lim_{n\to\infty}\left|\frac{(n+1)}{4(n-2)}\right| \overset{R.H}{=} \lim_{n\to\infty}\left|\frac{1}{4}\right| = \boxed{\frac{1}{4}} = \frac{1}{R} \qquad \Rightarrow R=4 \quad |x|<4$$

$$f'(x) = \frac{n^2(n-1)(n-2)}{4^n} x^{n-1} - \frac{n^2(n-1)(n-2)}{4^n x} x^n$$

$$a_n = \frac{n^2(n-1)(n-2)}{4^n x} \qquad a_{n+1} = \frac{(n+1)^2 n(n-1)}{4^{n+1} x}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 \cancel{n}\cancel{(n-1)}}{4^{n+1}\cancel{x}} \cdot \frac{4^n \cancel{x}}{n^{\cancel{2}}\cancel{(n-1)}(n-2)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2}{4n(n-2)}\right| = \lim_{n\to\infty}\left|\frac{n^2+2n+1}{4(n^2-2n)}\right| \overset{R.H.}{=} \lim_{n\to\infty}\left|\frac{2n+2}{8n-8}\right|$$

$$\overset{R.H.}{=} \lim_{n\to\infty}\left|\frac{2}{8}\right| = \boxed{\frac{1}{4}} = \frac{1}{R} \qquad \Rightarrow R=4 \quad |x|<4$$

Intervalle de convergence $-4<x<4$ ($R=4$ centré en $x=0$).

$\boxed{6.14}$ $(1-x)y'' - y' + xy = 0$

Σ $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$

$y'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots$

$y''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \dots$

① $(1-x)y'' = 2a_2 - 2a_2x + 6a_3x - 6a_3x^2 + 12a_4x^2 - 12a_4x^3 + \dots$

② $-y' = -a_1 - 2a_2x - 3a_3x^2 - 4a_4x^3 - \dots$

③ $xy = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \dots$

$Ly = ① + ② + ③ = 0$

$= (2a_2 - a_1) + (6a_3 - 4a_2 + a_0)x + (12a_4 - 9a_3 + a_1)x^2 + (-12a_4 - 4a_4 + a_2)x^3 + \dots$

$$\left.\begin{array}{l} x^0 \to 2a_2 - a_1 = 0 \\ x^1 \to 6a_3 - 4a_2 + a_0 = 0 \\ x^2 \to 12a_4 - 9a_3 + a_1 = 0 \\ x^3 \to -16a_4 + a_2 = 0 \end{array}\right\}$$

5 inconnues / 4 équations.
→ Solutions en fonction de a_0 & a_1

$$\boxed{a_2 = \frac{a_1}{2}}$$

$$a_3 = \frac{4a_2 - a_0}{6} = \frac{4\left(\frac{a_1}{2}\right)}{6} - \frac{a_0}{6} = \boxed{\frac{a_1}{3} - \frac{a_0}{6}}$$

$$a_4 = \frac{9a_3 + a_1}{12} = \frac{9\left(\frac{a_1}{3} - \frac{a_0}{6}\right)}{12} - \frac{a_1}{12} = \frac{a_1}{4} - \frac{a_0}{8} - \frac{a_1}{12} = \boxed{\frac{a_1}{6} - \frac{a_0}{8}}$$

$$\boxed{y(x) = a_0 + a_1 x + \left(\frac{a_1}{2}\right)x^2 + \left(\frac{a_1}{3} - \frac{a_0}{6}\right)x^3 + \left(\frac{a_1}{6} - \frac{a_0}{8}\right)x^4 + \ldots}$$

$\boxed{6.22}$ $P_4(0{,}7) \Rightarrow P_4(x)$ & $x = 0{,}7$

$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$, $n = 1, 2, \ldots$ $\Rightarrow P_4 \to n = 3$

$$(3+1)P_4(x) = (2(3)+1)xP_3(x) - 3P_2(x)$$
$$4P_4(0{,}7) = 7(0{,}7)P_3(0{,}7) - 3P_2(0{,}7)$$

1° $P_3(x) = \frac{1}{2}(5x^3 - 3x)$ $\quad P_2(x) = \frac{1}{2}(3x^2 - 1)$

$$P_4(0{,}7) = \frac{4{,}9}{8}\left(5(0{,}7)^3 - 3(0{,}7)\right) - \frac{3}{8}\left(3(0{,}7)^2 - 1\right)$$
$$= -0{,}4120625$$

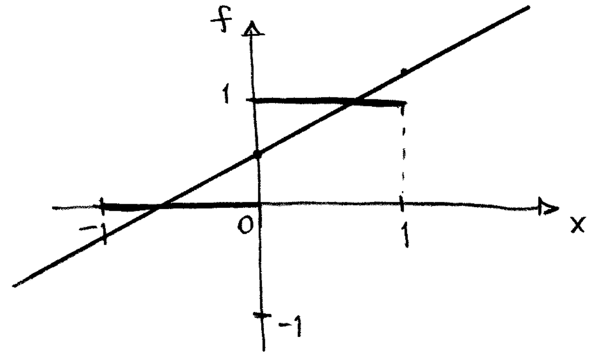
2° $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$$P_4(0{,}7) = \frac{1}{8}(35(0{,}7)^4 - 30(0{,}7)^2 + 3) = -0{,}4120625$$

⇒ Les 2 méthodes donnent le même résultat.

07.6

$\boxed{6.29}$ $f(x) = \begin{cases} 0 & , -1 < x < 0 \\ 1 & , 0 < x < 1 \end{cases}$



$f(x) = \sum_{m=0}^{\infty} a_m P_m(x), \; -1 < x < 1$

$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx$

$a_0 = \frac{1}{2} \int_{-1}^{1} f(x) P_0(x)\, dx = \frac{1}{2} \int_0^1 (1)(1)\, dx = \frac{1}{2} x \Big|_0^1 = \boxed{\frac{1}{2}}$

$a_1 = \frac{3}{2} \int_{-1}^{1} f(x) P_1(x)\, dx = \frac{3}{2} \int_0^1 (1)\, x\, dx = \frac{3}{2} \frac{x^2}{2} \Big|_0^1 = \boxed{\frac{3}{4}}$

$a_2 = \frac{5}{2} \int_{-1}^{1} f(x) P_2(x)\, dx = \frac{5}{2} \int_0^1 (1) \frac{1}{2} (3x^2 - 1)\, dx$

$= \frac{5}{4} \int_0^1 3x^2 - 1\, dx = \frac{5}{4} (x^3 - x) \Big|_0^1 = \boxed{0}$

$f(x) \approx \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) + 0\, P_2(x) = \boxed{\frac{1}{2} + \frac{3}{4} x}$ ✓

$\boxed{10.7}$ $y' = x + \sin y \; , \; y(0) = 0$ pour $0 \leq x \leq 1$ $\Rightarrow h = 0{,}1$

EULER AMÉLIORÉE

$x_n = x_0 + hn = 0 + 0{,}1n$ pour $n = 0, 1, 2 \ldots$

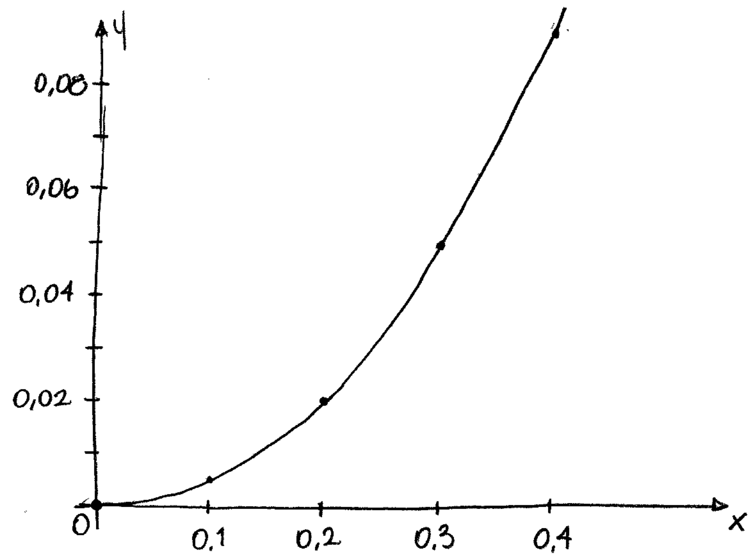
$y_0^c = 0$

$y_{n+1}^p = y_n^c + 0{,}1 (x_n + \sin y_n^c)$

$y_{n+1}^c = y_n^c + 0{,}05 ((x_n + \sin y_n^c) + (x_{n+1} + \sin y_{n+1}^p))$

Table : Résultats numériques

n	x_n	y_n^P	y_n^C
0	0	—	0,0000
1	0,1	0,0000	0,0050
2	0,2	0,0155	0,0210
3	0,3	0,0431	0,0492
4	0,4	0,0842	0,0909



10 | 1,0 | 0,69515 | 0,70592

$\boxed{10.12}$ $y' = x + \sin y$, $y(0) = 0$ pour $0 \leq x \leq 1$ $\Rightarrow h = 0,1$

RUNGE-KUTTA 4.

$x_n = x_0 + hn = 0 + 0,1n$ pour $n = 0, 1, 2 \ldots$

$y_0 = 0$

$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\Rightarrow k_1 = hf(x_n, y_n) = 0,1(x_n + \sin y_n)$

$k_2 = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1) = 0,1((x_n + 0,05) + \sin(y_n + \frac{1}{2}k_1))$

$k_3 = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2) = 0,1((x_n + 0,05) + \sin(y_n + \frac{1}{2}k_2))$

$k_4 = hf(x_n + h, y_n + k_3) = 0,1((x_n + 0,1) + \sin(y_n + k_3))$

Table : Résultats numériques

n	x_n	K_1	K_2	K_3	K_4	y_n
0	0	0,0000	0,0050	0,0052	0,0105	0,0000
1	0,1	0,0105	0,0160	0,0163	0,0221	0,0052
2	0,2	0,0221	0,0282	0,0286	0,0350	0,0214
3	0,3	0,0350	0,0417	0,0421	0,0492	0,0499
4	0,4					0,0918
10	1,0					0,70992