D6.1

MAT2784B - Devoir 6.

$\boxed{5.29}$ $F(s) = \dfrac{e^{-3s}}{s^2(s-1)} = e^{-3s}\left(\dfrac{1}{s^2(s-1)}\right)$

$$\mathcal{L}^{-1}(e^{-3s}F(s)) = u(t-a)f(t-a) = u(t-3)f(t-3)$$

$$F(s) = \frac{1}{s^2(s-1)} = \frac{As+B}{s^2} + \frac{C}{s-1}$$

$$1 = (As+B)(s-1) + Cs^2$$

$$1 = As^2 - As + Bs - B + Cs^2$$

$$1 = (A+C)s^2 + (B-A)s - B$$

$\Rightarrow A + C = 0$

$B - A = 0$

$\boxed{-B = 1}$ $\rightarrow -1 - A = 0$ $\boxed{A = -1}$ $\rightarrow 1 + C = 0$ $\boxed{C = 1}$

$$\Rightarrow F(s) = -\frac{1s-1}{s^2} + \frac{1}{s-1} = \frac{-1}{s} - \frac{1}{s^2} + \frac{1}{s-1}$$

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{-1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$

$$= -1 - \frac{t}{1!} + e^t = -1 - t + e^t = y(t)$$

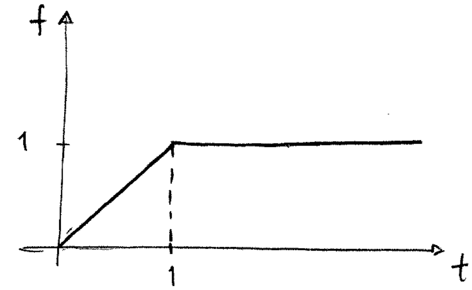
$$f(t-3) = -1 - (t-3) + e^{(t-3)} = 2 - t + e^{t-3}$$

Donc $\boxed{f(t) = \begin{cases} 0, & \text{si } 0 \leq t < 3 \\ 2 - t + e^{t-3} & \text{si } t \geq 3 \end{cases}}$

$$f(t) = -u(t-3) - u(t-3)[t-3] + u(t-3)e^{t-3}$$

Déf 2

$\boxed{5.32}$ $f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 1, & t \geq 1 \end{cases}$




Fonction d'Heaviside :

$f(t) = t - u(t-1)t + u(t-1)1$

$= t - u(t-1)(t-1)$

$F(s) = \mathcal{L}(t) - \mathcal{L}(u(t-1)f(t-1))(s)$

$= \frac{1}{s^2} - e^{-s}\mathcal{L}(t-1) \qquad = \frac{1}{s^2} - e^{-s}\frac{1}{s^2}$

$$\boxed{F(s) = \frac{1}{s^2}(1-e^{-s})}$$

✓

$\boxed{5.42}$ $y'' + y = \sin 3t \quad y(0) = 0 \;,\; y'(0) = 0$

$\mathcal{L}(y)(s) = Y(s)$

$\mathcal{L}(y'') + \mathcal{L}(y) = s^2 Y(s) - \cancel{sy(0)}^{\to 0} - \cancel{y'(0)}^{\to 0} + Y(s) = \mathcal{L}(\sin 3t)$

$s^2 Y(s) + Y(s) = \frac{3}{s^2+3^2}$

$(s^2+1)Y(s) = \frac{3}{s^2+9} \quad \Rightarrow \quad Y(s) = \frac{3}{(s^2+1)(s^2+9)}$

$\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+9} = 3$ (fonction paire)

$(As+B)(s^2+9) + (Cs+D)(s^2+1) = 3$

$As^3 + 9As + Bs^2 + 9B + Cs^3 + Cs + Ds^2 + D = 3$

$(A+C)s^3 + (B+D)s^2 + (9A+C)s + (9B+D) = 3$

$$\left.\begin{array}{l} ① \; A + C = 0 \\ ② \; B + D = 0 \\ ③ \; 9A + C = 0 \\ ④ \; 9B + D = 3 \end{array}\right\}$$

$④ - ② = 8B = 3 \Rightarrow \boxed{B = \frac{3}{8}}$

$② = \frac{3}{8} + D = 0 \Rightarrow \boxed{D = -\frac{3}{8}}$

} partie paire

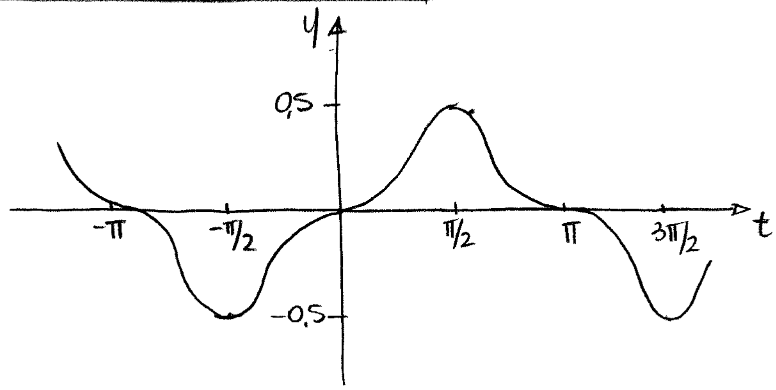
$\boxed{A \,\&\, C = 0}$ partie impaire

$$Y(s) = \frac{3/8}{s^2+1} - \frac{3/8}{s^2+9} = \frac{3}{8}\left(\frac{1}{s^2+1}\right) - \frac{1}{8}\left(\frac{3}{s^2+3^2}\right)$$

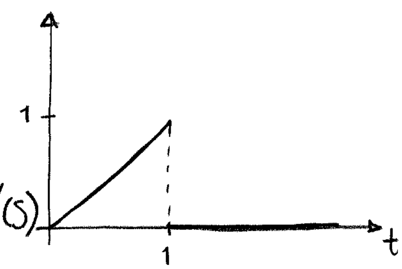
$$y(t) = \mathcal{L}^{-1}(Y(s)) = \frac{3}{8}\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{8}\mathcal{L}^{-1}\left(\frac{3}{s^2+3^2}\right)$$

$$\boxed{y(t) = \frac{3}{8}\sin t - \frac{1}{8}\sin 3t}$$

✓




$\boxed{5.49}$ $y'' - 5y' + 6y = \begin{cases} t, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$ $\quad y(0) = 0, \; y'(0) = 1$




$\mathcal{L}(y)(s) = Y(s)$

$\mathcal{L}(y'') - 5\mathcal{L}(y') + 6\mathcal{L}(y)$

$s^2 Y(s) - \cancel{s y(0)}^{0} - \cancel{y'(0)}^{1} - 5(sY(s) - \cancel{y(0)}^{0}) + 6Y(s)$

$(s^2 - 5s + 6)Y(s) - 1 = F(s)$

Fonction d'Heaviside :

$f(t) = \left\{\begin{matrix} t, & 0 \leq t \leq 1 \\ 0, & t \geq 1 \end{matrix}\right\} = t - u(t-1)t + \cancel{u(t-1)0}^{0}$

$f(t) = t - u(t-1)t = t - u(t-1)t - u(t-1) + u(t-1)$

$\quad = t - u(t-1)(t-1) - u(t-1)$

$\mathcal{L}(f(t)) = \mathcal{L}(t) - \mathcal{L}(u(t-1)(t-1) + u(t-1))$

$\quad = \frac{1}{s^2} - e^{-s}\mathcal{L}(t+1)$

$\quad = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$

$$\Rightarrow (s-2)(s-3)Y(s) - 1 = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$$

$$(s-2)(s-3)Y(s) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right) + 1$$

$$Y(s) = \underbrace{\frac{1}{(s-2)(s-3)s^2}}_{①} - \underbrace{\frac{e^{-s}}{(s-2)(s-3)s^2}}_{②} - \underbrace{\frac{e^{-s}}{(s-2)(s-3)s}}_{③} + \underbrace{\frac{1}{(s-2)(s-3)}}_{④}$$

① $\frac{A}{s-2} + \frac{B}{s-3} + \frac{Cs+D}{s^2} = 1$

$(As-3A)s^2 + (Bs-2B)s^2 + (Cs+D)(s^2-5s+6) = 1$

$As^3 - 3As^2 + Bs^3 - 2Bs^2 + Cs^3 - 5Cs^2 + 6Cs + Ds^2 - 5Ds + 6D = 1$

$(A+B+C)s^3 + (-3A-2B-5C+D)s^2 + (6C-5D)s + 6D = 1$

$\Rightarrow A+B+C = 0$

$-3A-2B-5C+D = 0$

$6C-5D = 0$

$6D = 1 \Rightarrow \boxed{D = \frac{1}{6}}$

$6C - 5(\frac{1}{6}) = 0 \Rightarrow \boxed{C = \frac{5}{36}}$

$-3A - 2B = 25/36 - 1/6 = 19/36$

$+2(A+B = -5/36)$

$-A = 9/36 \Rightarrow \boxed{A = -\frac{1}{4}}$

$(-1/4) + B + (5/36) = 0 \Rightarrow \boxed{B = \frac{1}{9}}$

$$\Rightarrow -\frac{1}{4}\left(\frac{1}{(s-2)}\right) + \frac{1}{9}\left(\frac{1}{(s-3)}\right) + \frac{(5/36)s + (1/6)}{s^2}$$

② $\Rightarrow -e^{-s}F(s) = -e^{-s}\left(-\frac{1}{4}\left(\frac{1}{s-2}\right) + \frac{1}{9}\left(\frac{1}{s-3}\right) + \frac{5}{36}\left(\frac{1}{s}\right) + \frac{1}{6}\left(\frac{1}{s^2}\right)\right)$

③ $\Rightarrow -e^{-s}F(s)$

$F(s) = \frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{s} = 1$

$As^2 - 3As + Bs^2 - 2Bs + Cs^2 - 5Cs + 6C = 1$

$(A+B+C)As^2 + (-3A-2B-5C)s + 6C = 1$

D 6.5

$$\Rightarrow A+B+C=0$$
$$-3A-2B-5C=0$$
$$6C=1 \Rightarrow \boxed{C=\frac{1}{6}}$$

$$-3A-2B=5/6$$
$$+2(A+B=-1/6)$$
$$-A=1/2 \Rightarrow \boxed{A=\frac{-1}{2}}$$

$$(-1/2)+B+(1/6)=0 \Rightarrow \boxed{B=\frac{1}{3}}$$

$$-e^{-s}\left(-\frac{1}{2}\left(\frac{1}{s-2}\right)+\frac{1}{3}\left(\frac{1}{s-3}\right)+\frac{1}{6}\left(\frac{1}{s}\right)\right)$$

④ $$\frac{A}{s-2}+\frac{B}{s-3}=1$$

$$As-3A+Bs-2B=1$$
$$A+B=0$$
$$-3A-2B=1$$

$$-3A-2B=1$$
$$+2(A+B=0)$$
$$-A=1 \Rightarrow \boxed{A=-1} \Rightarrow \boxed{B=1}$$

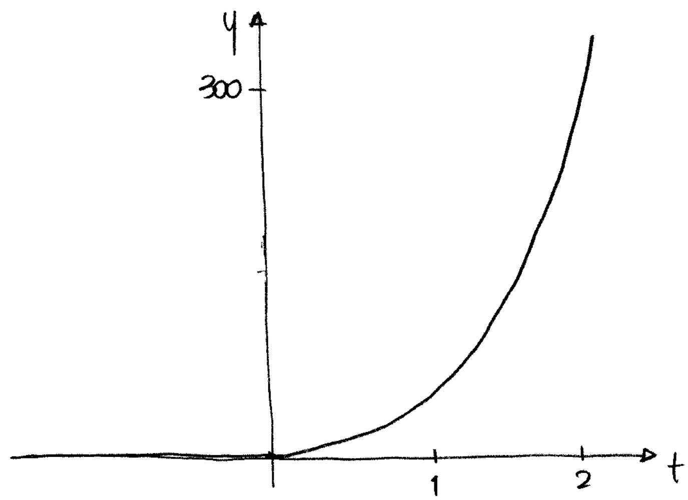
$$\Rightarrow -\frac{1}{s-2}+\frac{1}{s-3}$$

$$Y(s)=\frac{-1}{4}\left(\frac{1}{s-2}\right)+\frac{1}{9}\left(\frac{1}{s-3}\right)+\frac{5}{36}\left(\frac{1}{s}\right)+\frac{1}{6}\left(\frac{1}{s^2}\right)-\frac{1}{s-2}+\frac{1}{s-3}$$
$$-e^{-s}\left[\frac{-1}{4}\left(\frac{1}{s-2}\right)+\frac{1}{9}\left(\frac{1}{s-3}\right)+\frac{5}{36}\left(\frac{1}{s}\right)+\frac{1}{6}\left(\frac{1}{s^2}\right)-\frac{1}{2}\left(\frac{1}{s-2}\right)+\frac{1}{3}\left(\frac{1}{s-3}\right)+\frac{1}{6}\left(\frac{1}{s}\right)\right]$$

$$Y(s)=\frac{5}{36}\left(\frac{1}{s}\right)+\frac{1}{6}\left(\frac{1}{s^2}\right)-\frac{5}{4}\left(\frac{1}{s-2}\right)+\frac{10}{9}\left(\frac{1}{s-3}\right)-e^{-s}\left[\frac{11}{36}\left(\frac{1}{s}\right)+\frac{1}{6}\left(\frac{1}{s^2}\right)-\frac{3}{4}\left(\frac{1}{s-2}\right)+\frac{4}{9}\left(\frac{1}{s-3}\right)\right.$$

$$\mathcal{L}^{-1}(Y(s))=\boxed{y(t)=\frac{5}{36}+\frac{t}{6}-\frac{5}{4}e^{2t}+\frac{10}{9}e^{3t}-u(t-1)\left[\frac{11}{36}+\frac{(t-1)}{6}-\frac{3}{4}e^{2t-2}+\frac{4}{9}e^{3t-3}\right.}$$

D 6, 6




$\boxed{5.52}$ $y'' + 3y' + 2y = 1 - u(t-1) \quad y(0) = 0 \;,\; y'(0) = 1$

$\mathcal{L}(y)(s) = Y(s)$

$\mathcal{L}(y'') + 3\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(1) - \mathcal{L}(u(t-1))$

$s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{1} + 3(sY(s) - \cancel{y(0)}^{0}) + 2Y(s) = 1/s - e^{-s}\, 1/s$

$(s^2 + 3s + 2)Y(s) - 1 = \frac{1}{s}(1 - e^{-s})$

$(s+1)(s+2)Y(s) = \frac{1}{s}(1 - e^{-s}) + 1$

$$Y(s) = \underset{①}{\frac{1}{(s+1)(s+2)s}}(1 - e^{-s}) + \underset{②}{\frac{1}{(s+1)(s+2)}}$$

① $\frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s} = 1$

$As^2 + 2As + Bs^2 + Bs + Cs^2 + 3Cs + 2C = 1$

$(A + B + C) = 0$

$(2A + B + 3C) = 0$

$2C = 1 \Rightarrow \boxed{C = \frac{1}{2}}$

$2A + B = -3/2$

$-\,(A + B = -1/2)$

$\boxed{A = -1}$

$-1 + B + (1/2) = 0 \Rightarrow \boxed{B = 1/2}$

$\Rightarrow \frac{-1}{s+1} + \frac{1/2}{s+2} + \frac{1/2}{s}$

② $\frac{A}{s+1} + \frac{B}{s+2} = 1$

$As + 2A + Bs + B = 1$

$$\left.\begin{array}{l} A + B = 0 \\ 2A + B = 1 \end{array}\right\} \quad \begin{array}{r} 2A + B - 1 \\ -(A + B = 0) \\ \hline \boxed{A = 1} \end{array} \quad \Rightarrow \boxed{B = -1}$$

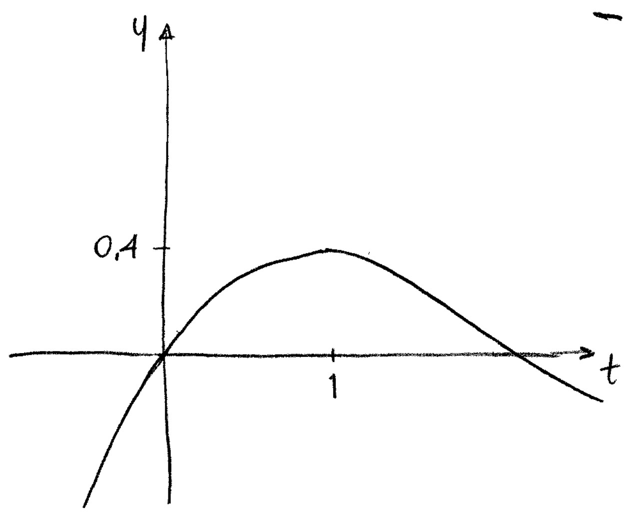
$\Rightarrow \frac{1}{s+1} - \frac{1}{s+2}$

$$Y(s) = \cancel{\frac{-1}{s+1}} + \frac{1}{2}\left(\frac{1}{s+2}\right) + \frac{1}{2}\left(\frac{1}{s}\right) - e^{-s}\left(\frac{-1}{s+1} + \frac{1}{2}\left(\frac{1}{s+2}\right) + \frac{1}{2}\left(\frac{1}{s}\right)\right)$$
$$+ \cancel{\frac{1}{s+1}} - \frac{1}{s+2}$$

$$Y(s) = \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s+2}\right) - e^{-s}\left(\frac{-1}{s+1} + \frac{1}{2}\left(\frac{1}{s+2}\right) + \frac{1}{2}\left(\frac{1}{s}\right)\right)$$

$$\mathcal{L}^{-1}(Y(s)) = y(t) = \frac{1}{2} - \frac{e^{-2t}}{2} - u(t-1)\left[-e^{-t+1} + \frac{e^{-2t+2}}{2} + \frac{1}{2}\right]$$

mikeert $-e^{-(t-1)}$ $\quad \frac{e^{-2(t-1)}}{2}$



D 6, 8

$\boxed{5.53}$ $y'' - y = \sin t + \delta(t - \pi/2) \quad y(0) = 3{,}5 \;,\; y'(0) = -3{,}5$

$$\mathcal{L}(y)(s) = Y(s)$$
$$\mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(\sin t) + \mathcal{L}(\delta(t - \pi/2))$$
$$s^2 Y(s) - s y(0) - y'(0) - Y(s) = \frac{1}{s^2+1} + e^{-(\pi/2)s}$$
$$(s^2 - 1)Y(s) - 3{,}5s + 3{,}5 = \frac{1}{s^2+1} + e^{-(\pi/2)s}$$
$$Y(s) = \frac{3{,}5s}{s^2-1} - \frac{3{,}5}{s^2-1} + \underset{①}{\frac{1}{(s^2-1)(s^2+1)}} + \frac{e^{-(\pi/2)s}}{s^2-1}$$

① $\dfrac{As+B}{s^2-1} + \dfrac{Cs+D}{s^2+1} = 1$ *(fonction paire)*

$$As^3 + As + Bs^2 + B + Cs^3 - Cs + Ds^2 - D = 1$$
$$(A+C)s^3 + (B+D)s^2 + (A-C)s + (B-D) = 1$$

$$\left.\begin{array}{l} A+C=0 \\ B+D=0 \\ A-C=0 \\ B-D=1 \end{array}\right\} \quad \begin{array}{r} B-D=1 \\ +\;B+C=0 \\ \hline 2B=1 \end{array} \Rightarrow \boxed{B = \frac{1}{2}} \rightarrow \boxed{D = -\frac{1}{2}}$$ *partie paire*

$\boxed{A = C = 0}$ *partie impaire*

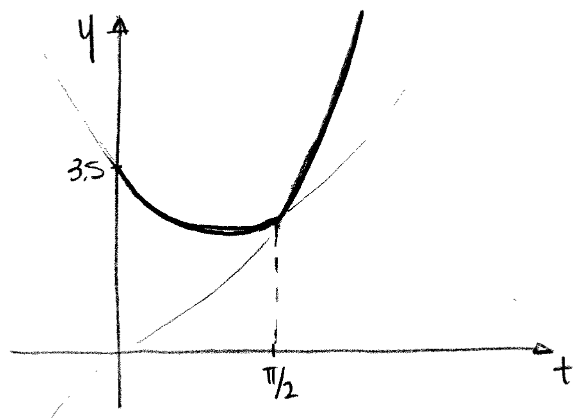
$$\Rightarrow \frac{1/2}{s^2-1} - \frac{1/2}{s^2+1}$$

$$Y(s) = 3.5\left(\frac{s}{s^2-1}\right) - 3{,}5\left(\frac{1}{s^2-1}\right) + \frac{1}{2}\left(\frac{1}{s^2-1}\right) - \frac{1}{2}\left(\frac{1}{s^2+1}\right) + e^{-(\pi/2)s}\left(\frac{1}{s^2-1}\right)$$

$$\mathcal{L}^{-1}(Y(s)) = y(t) = 3{,}5\cosh t - 3{,}5\sinh t + \frac{1}{2}\sinh t - \frac{1}{2}\sin t$$
$$+\, u(t - \pi/2)[\sinh(t - \pi/2)]$$

$$\boxed{y(t) = 3{,}5\cosh t - 3\sinh t \cancel{+} - \frac{\sin t}{2} + u(t - \pi/2)[\sinh(t - \pi/2)]}$$

✓




**9.10** $\int_0^2 \frac{1}{x+4}\,dx \rightarrow 10^{-5}$

1) Methode des trapèzes (composés)

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + f(x_n)\right] - \frac{(b-a)h^2}{12} f''(\xi)$$

$$f(x) = \frac{1}{x+4} \rightarrow f'(x) = -\frac{1}{(x+4)^2}$$

$$f''(x) = \frac{2}{(x+4)^3}$$

$$f''(0) = \frac{1}{32} > f''(2) = \frac{1}{108}$$

$$M = \left| \max_{0 \leq x \leq 2} f''(x) \right| = \frac{1}{32}$$

$$\left| \frac{(b-a)h^2}{12} f''(\xi) \right| \leq \frac{2h^2}{12} M = \frac{h^2}{192} < 10^{-5}$$

$$h \leq \sqrt{192 \times 10^{-5}} = 0{,}043818 \Rightarrow \frac{2}{h} = 45{,}6435 \leq n = 46$$

Alors $\boxed{h = \frac{2}{46} \ \& \ n = 46}$

2) Méthode de Simpson (composée)

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + f(x_{2m})\right]$$
$$-\frac{(b-a)h^4}{180} f^{(iv)}(\xi).$$

$$f(x) = \frac{1}{x+4} \quad \rightarrow \quad f'''(x) = \frac{-6}{(x+4)^4}$$
$$f^{(iv)}(x) = \frac{24}{(x+4)^5}$$

$$f^{(iv)}(0) = \frac{3}{128} \qquad f^{(iv)}(2) = \frac{1}{324} \qquad \rightarrow \quad \frac{1}{324} \leq f^{(4)}(x) \leq \frac{3}{128}$$

$$M = \left|\max_{0 \leq x \leq 2} f^{(iv)}(x)\right| = \frac{3}{128}$$

$$\left|\frac{(b-a)h^4}{180} f^{(iv)}(\xi)\right| \leq \frac{2h^4}{180} M = \frac{h^4}{3840} < 10^{-5}$$

$$h < \sqrt[4]{3840 \times 10^{-5}} = 0{,}442673 \Rightarrow \frac{2}{h} = 4{,}5180 < 2m = n = 6$$

Alors $\boxed{h = \frac{2}{6} \ \& \ n = 6}$

3) Méthode des points milieu (composés)

$$\int_a^b f(x)\,dx = h\left[f(x_1^*) + f(x_2^*) + \dots + f(x_n^*)\right] + \frac{(b-a)h^2}{24} f''(\xi)$$

$$\frac{1}{108} < f''(x) \leq \frac{1}{32} \quad \Rightarrow \quad M = \left|\max_{0 \leq x \leq 2} f''(x)\right| = \frac{1}{32}$$

$$\left|\frac{(b-a)h^2}{24} f''(\xi)\right| \leq \frac{2h^2}{24} M = \frac{h^2}{384} \leq 10^{-5}$$

$h < \sqrt{384 \times 10^{-5}} = 0{,}061968 \Rightarrow \frac{2}{h} = 32{,}2749 \leq n = 33$

Alors $\boxed{h = \frac{2}{33} \ \& \ n = 33}$

$\boxed{9.11}$ $\int_1^{1,5} x^2 \ln x \, dx \Rightarrow$ Calculer $R_{3.3}$

Intégration de Romberg : $h_k = \frac{h}{2^{k-1}}$

$h = 1{,}5 - 1 = 0{,}5$

$\Rightarrow h_1 = h = 0{,}5 \quad h_2 = \frac{h}{2} = 0{,}25 \quad h_3 = \frac{h}{2^2} = 0{,}125$

Avec la méthode du trapèze & $h_1 = 0{,}5$

$R_{1.1} = \frac{h_1}{2}[f(1) + f(1{,}5)] = \frac{0{,}5}{2}[1^2 \ln(1) + (1{,}5)^2 \ln(1{,}5)]$

$= 0{,}2280741233$

Avec $h_2 = 0{,}25$

$R_{2.1} = \frac{h_2}{2}[f(1) + 2f(1{,}25) + f(1{,}5)] = 0{,}2012025114$

Avec $h_3 = 0{,}125$

$R_{3.1} = \frac{h_3}{2}[f(1) + 2f(1{,}125) + 2f(1{,}25) + 2f(1{,}375) + f(1{,}5)]$

$= 0{,}1944944732$

Table d'intégration de Romberg:

| | j = 1 | j = 2 | j = 3 |
|---|---|---|---|
| k = 1 | 0,2280741233 | | |
| 2 | 0,2012025114 | 0.1922453074 | |
| 3 | 0,1944944732 | 0,1922584604 | 0.1922593373 |

$$R_{2,2} = R_{2,1} + \frac{R_{2,1} - R_{1,1}}{4^1 - 1} = 0,1922453074$$

$$R_{3,2} = R_{3,1} + \frac{R_{3,1} - R_{2,1}}{4^1 - 1} = 0,1922584604$$

$$R_{3,3} = R_{3,2} + \frac{R_{3,2} - R_{2,2}}{4^2 - 1} = 0,1922593373$$

Alors $\boxed{I \approx R_{3,3} = 0,1922593373}$