2011/03/02
MAT 2784

Devoir 5

Question 3.22:

$$y'' + 3y' + 2y = 5e^{-2x}$$

On trouve d'abord la solution à l'équation homogène.

$$y'' + 3y' + 2y = 0 \qquad \text{On pose} \quad y = e^{\lambda x}$$

$$\lambda^2 + 3\lambda + 2 = 0$$
$$(\lambda + 2)(\lambda + 1) = 0 \quad \Rightarrow \lambda_1 = -2 \qquad \lambda_2 = -1$$

$$y_h(x) = c_1 e^{-2x} + c_2 e^{-x}$$

Donc une solution particulière est de la forme

$$y_p(x) = axe^{-2x}$$

$$y_p'(x) = a\left[-2xe^{-2x} + e^{-2x}\right]$$

$$y_p''(x) = a\left[4xe^{-2x} - 2e^{-2x} - 2e^{-2x}\right] = a\left[4xe^{-2x} - 4e^{-2x}\right]$$

$$y_p'' + 3y_p' + 2y = a\left[4xe^{-2x} - 4e^{-2x} - 6xe^{-2x} + 3e^{-2x} + 2xe^{-2x}\right]$$

$$= a\left[-e^{-2x}\right] = 5e^{-2x} \qquad \Rightarrow a = -5$$

$$y_g(x) = y_h(x) + y_p(x)$$

$$\therefore \; y_g(x) = c_1 e^{-2x} + c_2 e^{-x} - 5xe^{-2x} \qquad \text{(solution générale)}$$

✓

Question 3.28:

$y''' + y' = x \qquad y(0) = 0 \qquad y'(0) = 1 \qquad y''(0) = 0$

On trouve d'abord la solution à l'équation homogène

$\lambda^3 + \lambda = 0 \quad \Rightarrow \quad \lambda(\lambda^2 + 1) = 0$

$\Rightarrow \lambda_1 = 0 \qquad \lambda_2 = i \qquad \lambda_3 = -i$

$\Rightarrow y_h(x) = c_1 + c_2 \cos x + c_3 \sin x$

Une solution particulière est de la forme

$y_p(x) = ax + bx^2$

$y'_p(x) = a + 2bx \qquad y'''_p(x) = 0$

$y'''_p + y'_p = a + 2bx = x \qquad \Rightarrow a = 0 \qquad b = 1/2$

$\therefore y_g(x) = c_1 + c_2 \cos x + c_3 \sin x + \frac{x^2}{2}$ (solution générale)

$y(0) = 0 = c_1 + c_2$

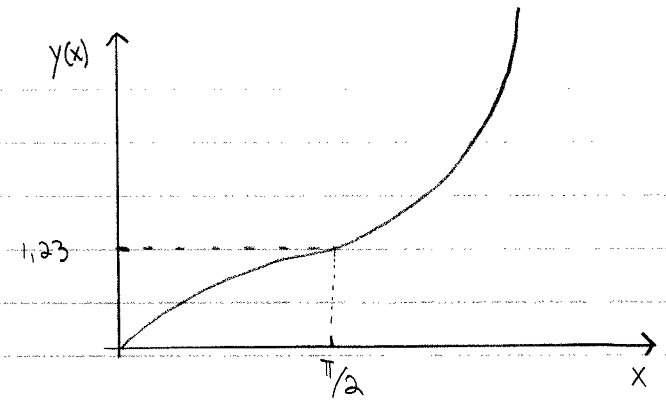
$y'(x) = -c_2 \sin x + c_3 \cos x + x$

$y'(0) = 1 = c_3$

$y''(x) = -c_2 \cos x - c_3 \sin x + 1$

$y''(0) = 0 = -c_2 + 1 \quad \Rightarrow \quad c_2 = 1 \quad \Rightarrow \quad c_1 = -1$

$\therefore y(x) = -1 + \cos x + \sin x + \frac{x^2}{2}$ (solution unique)



Question 3.32

$$y'' + 6y' + 9y = \frac{e^{-3x}}{x^3}$$

On résout d'abord l'équation homogène

$$y'' + 6y' + 9y = 0$$

$$\lambda^2 + 6\lambda + 9 = 0$$
$$(\lambda + 3)^2 = 0 \qquad \Rightarrow \lambda_{1,2} = -3$$

$$y_h(x) = c_1 e^{-3x} + c_2 x e^{-3x}$$

On trouve une solution particulière par variation des paramètres

$$y_p(x) = C_1(x)\, e^{-3x} + C_2(x)\, x e^{-3x}$$

$$\begin{bmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & -3xe^{-3x} + e^{-3x} \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ x^{-3} e^{-3x} \end{bmatrix}$$

$$\left[\begin{array}{cc|c} e^{-3x} & xe^{-3x} & 0 \\ -3e^{-3x} & -3xe^{-3x} + e^{-3x} & x^{-3} e^{-3x} \end{array}\right] \sim \left[\begin{array}{cc|c} e^{-3x} & xe^{-3x} & 0 \\ 0 & e^{-3x} & x^{-3} e^{-3x} \end{array}\right]$$

on multiplie partout par e^{3x}

$$c_2' e^{-3x} = x^{-3} e^{-3x} \Rightarrow c_2' = x^{-3}$$

$$c_1' e^{-3x} + c_2' x e^{-3x} = 0$$
$$c_1' e^{-3x} + x^{-2} e^{-3x} = 0$$
$$c_1' e^{-3x} = -x^{-2} e^{-3x}$$
$$c_1' = -x^{-2}$$

$$\Rightarrow c_1 = \int -x^{-2}\, dx = x^{-1} \qquad c_2 = \int x^{-3}\, dx = -\frac{x^{-2}}{2}$$

$$\Rightarrow y_p(x) = x^{-1}e^{-3x} - \frac{x^{-2}}{2} \cdot x e^{-3x} = x^{-1}e^{-3x} - \frac{x^{-1}e^{-3x}}{2} = \frac{x^{-1}e^{-3x}}{2}$$

$$y_g(x) = y_h(x) + y_p(x)$$

$$y_g(x) = Ae^{-3x} + Bxe^{-3x} + \frac{e^{-3x}}{2x}$$ ✓

Question 5.6:

$$f(t) = \sin^2 t = \frac{1}{2} - \frac{\cos(2t)}{2}$$

$$\mathcal{L}(\sin^2 t)(s) = \int_0^\infty e^{-st} \sin^2 t\, dt = \int_0^\infty e^{-st} \cdot \frac{1-\cos 2t}{2}\, dt$$

$$= \int_0^\infty \frac{e^{-st}}{2} dt - \int_0^\infty \frac{\cos 2t}{2} dt = \frac{1}{2s} - \frac{1}{2} \cdot \frac{s}{s^2+2^2}$$

$$= \frac{1}{2} \cdot \frac{s^2+4-s^2}{s(s^2+4)} = \frac{4}{2s(s^2+4)} = \frac{2}{s(s^2+4)}$$ ✓

DIRECT:

$$\frac{1}{2} - \frac{\cos(2t)}{2} \xrightarrow{\mathcal{L}} \frac{1}{2s} - \frac{1}{2} \cdot \frac{s}{s^2+4}$$ ✓

DS.5

Question 5.18

$$F(s) = \frac{4}{s^2-9} = \frac{4}{s^2-3^2} = 4 \cdot \frac{1}{s^2-3^2} = 4 \cdot \mathcal{L}\left(\frac{1}{3}\sinh 3t\right)$$

$$\Rightarrow \mathcal{L}^{-1}(F(s)) = \frac{4}{3}\sinh 3t$$

✓

Question 5.44

$y'' + y = t \qquad y(0) = 0 \qquad y'(0) = 0$

On applique la transformation de Laplace de chaque côté et on obtient

$$s^2\mathcal{L}(y) - s\,y(0) - y'(0) + \mathcal{L}(y) = \frac{1}{s^2}$$

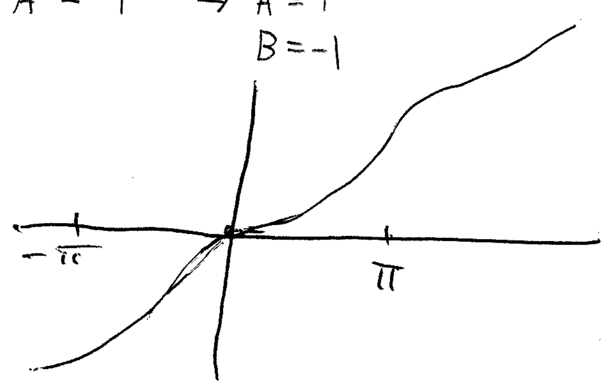
$$(s^2+1)\mathcal{L}(y) - 0 - 0 = \frac{1}{s^2}$$

$$\mathcal{L}(y) = \frac{1}{s^2(s^2+1)} = \frac{A}{s^2} + \frac{B}{s^2+1}$$

$$\Rightarrow As^2 + A + Bs^2 = (A+B)s^2 + A = 1 \quad \Rightarrow A = 1,\ B = -1$$

$$\mathcal{L}(y) = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$\Rightarrow y(t) = t - \sin t$$



✓

tracer la solution ? (−1)

Question 9.1

a) $dfn = \frac{1}{2 \cdot 0{,}1}\left[f(1{,}2+0{,}1) - f(1{,}2-0{,}1)\right]$

$= \frac{1}{0{,}2}\left[f(1{,}3) - f(1{,}1)\right]$

$= \frac{1}{0{,}2}\left[0{,}27253179303401 - 0{,}33287108369808\right]$

$= -0{,}3016964533$

b) $f'(x) = \sinh x - \cosh x$

$f'(1{,}2) = -0{,}301194212 = dfe$

c) $\varepsilon = dfn - dfe = -0{,}3016964533 - (-0{,}301194212)$

$= -5{,}02241 \cdot 10^{-4}$

d) $f^{(3)}(x) = \sinh x - \cosh x = \frac{e^x - e^{-x}}{2} - \frac{e^x + e^{-x}}{2} = -e^{-x}$

$|-e^{-x}|$ est décroissante sur $[1{,}1 ; 1{,}3]$ donc

$f^{(3)}(\xi) \leq \max\limits_{1{,}1 \leq x \leq 1{,}3} |-e^{-x}| = e^{-1{,}1} = 0{,}3328710837$

Erreur de méthode $= \frac{0{,}1^2}{6} \cdot 0{,}3328710837 = 5{,}54785 \cdot 10^{-4} > |\varepsilon|$

$|\varepsilon|$ est donc borné par le module de l'erreur de méthode. ✓

Question 9.2

$$N_1(0,4) = N(0,4) = \frac{1}{2\cdot 0,4}\left[f(1,4+0,4) - f(1,4-0,4)\right] = 0,7451795937$$

$$N_1(0,2) = N(0,2) = \frac{1}{0,4}\left[f(1,6) - f(1,2)\right] = 0,7662613488$$

$$N_1(0,1) = N(0,1) = \frac{1}{0,2}\left[f(1,5) - f(1,3)\right] = 0,7715597060$$ ✓

$$N_2(0,4) = N_1(0,2) + \frac{N_1(0,2) - N_1(0,4)}{3} = 0,7732886005$$

$$N_2(0,2) = N_1(0,1) + \frac{N_1(0,1) - N_1(0,2)}{3} = 0,7733258251$$

$$N_3(0,4) = N_2(0,2) + \frac{N_2(0,2) - N_2(0,4)}{15} = 0,7733283067$$

Donc $f'(1,4) \approx 0,7733283067$ ✓