MAT2784-B - Devoir #4

1. $\boxed{4.1}$ $x^2y'' - 7xy' + 16y = 0$, $x > 0$ $\Rightarrow Ly := x^2y'' + axy' + by = 0$ Équation d'Euler-Cauchy

$$m_{1,2} = \frac{1-a}{2} \pm \frac{1}{2}\sqrt{(a-1)^2 - 4b}$$

$$= \frac{1-(-7)}{2} \pm \frac{1}{2}\sqrt{((-7)-1)^2 - 4(16)} = 4 \pm 0$$

$\boxed{m_1 = 4 = m_2}$ $\Rightarrow y_1(x) = x^m = x^4$

$y_2(x) = \ln(x)x^m = \ln(x)x^4$

Solution générale : $\boxed{y(x) = c_1x^4 + c_2(\ln x)x^4}$ ✓

2. $\boxed{4.2}$ $x^2y'' + 3xy' + y = 0$, $x > 0$, $y(1) = -1$, $y'(1) = 1$

$\Rightarrow Ly := x^2y'' + axy' + by = 0$ (Équation d'Euler-Cauchy).

$$m_{1,2} = \frac{1-a}{2} \pm \frac{1}{2}\sqrt{(a-1)^2 - 4b} = \frac{1-3}{2} \pm \frac{1}{2}\sqrt{(3-1)^2 - 4(1)} = -1 \pm 0$$

$\boxed{m_1 = -1 = m_2}$ $\Rightarrow y_1(x) = x^m = x^{-1}$

$y_2(x) = \ln(x)x^m = \ln(x)x^{-1}$

Solution générale : $\boxed{y(x) = c_1\frac{1}{x} + c_2\frac{(\ln x)}{x}}$

$$y(1) = c_1\frac{1}{(1)} + c_2\cancel{\frac{(\ln(1))}{1}}^{\to 0} = -1$$

$\boxed{c_1 = -1}$

$$y'(x) = -c_1\frac{1}{x^2} + c_2\left(\frac{1}{x^2} - \frac{\ln(x)}{x^2}\right)$$

$$y'(1) = -(-1)\frac{1}{(1)^2} + c_2\left(\frac{1}{(1)^2} - \frac{\cancel{\ln(1)}^{\,0}}{\cancel{(1)^2}}\right) = 1$$

$$1 + c_2 = 1$$

$$\boxed{c_2 = 0}$$

Solution unique : $\boxed{y(x) = \frac{-1}{x}}$

✓

3. $\boxed{4.3}$ $y''' + 3y'' - 4y' - 12y = 0$

$\Rightarrow y''' + a_2 y'' + a_1 y' + a_0 y = 0$ & $y = e^{\lambda x}$

Équation charactéristique : $p(\lambda) := \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$

$:= \lambda^3 + 3\lambda^2 - 4\lambda - 12 = 0$

$:= (\lambda - 2)(\lambda + 2)(\lambda + 3) = 0$

$y_1(x) = e^{2x}$, $y_2(x) = e^{-2x}$, $y_3(x) = e^{-3x}$

Solution générale : $\boxed{y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{-3x}}$

✓

4. $\boxed{4.4}$ $y''' + 12y'' + 36y' = 0$, $y(0) = 0$, $y'(0) = 1$, $y''(0) = -7$

Équation charactéristique : $\lambda^3 + 12\lambda^2 + 36\lambda = 0$

$\lambda(\lambda^2 + 12\lambda + 36) = 0$

$\lambda(\lambda + 6)^2 = 0$

$y_1(x) = e^{0x} = 1$, $\lambda_2 = \lambda_3 = -6 \Rightarrow y_2(x) = e^{-6x}$ & $y_3(x) = xe^{-6x}$

Solution générale : $\boxed{y(x) = c_1 + c_2 e^{-6x} + c_3 xe^{-6x}}$

$y'(x) = -6c_2 e^{-6x} + c_3(e^{-6x} - 6xe^{-6x})$

$y''(x) = 36c_2 e^{-6x} - 12c_3 e^{-6x} + 36c_3 xe^{-6x}$

$$W(x_0)\begin{bmatrix} 1 & e^{-6x} & xe^{-6x} \\ 0 & -6e^{-6x} & e^{-6x}-6xe^{-6} \\ 0 & 36e^{-6x} & -12e^{-6x}+36xe^{-6} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix}$$

X

en $x=0$

$$W(0) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -6 & 1 \\ 0 & 36 & -12 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix} \quad L_3 + 6L_2 \to L_3$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -6 & 1 \\ 0 & 0 & -6 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$-6C_3 = -1$

$-6C_2 + (1/6) = 1$

$C_1 + (-5/36) = 0$

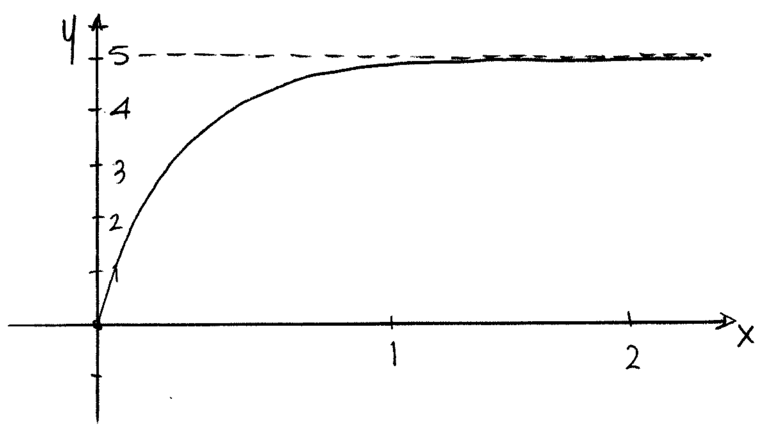
$$\boxed{\begin{aligned} C_3 &= 1/6 \\ C_2 &= -5/36 \\ C_1 &= 5/36 \end{aligned}}$$

<u>Solution unique</u> : $y(x) = \frac{5}{36} - \frac{5}{36}e^{-6x} + \frac{1}{6}xe^{-6x}$

$$\boxed{y(\cancel{x}) = \cancel{5} - 5\cancel{e^{-6x}} + \cancel{6}x\cancel{e^{-6x}}}$$

X (−2)

Ne satisfait pas $y(0)=0$!!



5 $\boxed{4.5}$ $y_1(x) = e^x$, $y_2(x) = e^{-x}$, $y_3(x) = \cosh x$

$$W(y_1, y_2, y_3)(x) = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{bmatrix} \end{array}$$

2 lignes parallèles $\Rightarrow W = 0 \Rightarrow$ lin. dép.

e^x, e^{-x} et $\cosh x$ sont 3 fois continûment dérivables, donc sol. d'une même éq. diff. par le cor. 3.2 à la page 50.

6. $\boxed{4.6}$ x, $x\ln x$, $x^2 \ln x$ $\quad e^{-2} < x < +\infty$

$$W(x) = \begin{vmatrix} x & x\ln x & x^2 \ln x \\ 1 & \ln x + 1 & 2x\ln x + x \\ 0 & 1/x & 2\ln x + 3 \end{vmatrix} = x \begin{vmatrix} 1 & \ln x & x\ln x \\ 1 & \ln x + 1 & 2x\ln x + x \\ 0 & 1/x & 2\ln x + 3 \end{vmatrix} \; L_2 - L_1 \to$$

$$= x \begin{vmatrix} 1 & \ln x & x\ln x \\ 0 & 1 & x\ln x + x \\ 0 & 1/x & 2\ln x + 3 \end{vmatrix} = x \begin{vmatrix} 1 & x\ln x + x \\ 1/x & 2\ln x + 3 \end{vmatrix}$$

$$= x(2\ln x + 3 - (\ln x + 1))$$

$$= x(\ln x + 2) \; \boxed{\neq 0}$$

Les fonctions sont linéairement <u>indépendantes</u>

$x, x\ln x, x^2 \ln x \in C^3[x > 0]$
donc sol. d'une même éq. diff.
par le cor. 3.2 à la page 50!

$\boxed{4.7}$ $f(8.1) = 16{,}94410$, $f(8.3) = 17.56492$, $f(8.6) = 18{,}50515$
$f(8.7) = 18{,}82091$. $\Rightarrow$ Interpoler $f(8.4)$
(Polynôme de Lagrange, Degré 1,2 & 3).

DEGRÉ 3

$$L_0(x) = \frac{(x-8{,}3)(x-8{,}6)(x-8{,}7)}{(8{,}1-8{,}3)(8{,}1-8{,}6)(8{,}1-8{,}7)} \Rightarrow L_0(8{,}4) = -0{,}1$$

$$L_1(x) = \frac{(x-8{,}1)(x-8{,}6)(x-8{,}7)}{(8{,}3-8{,}1)(8{,}3-8{,}6)(8{,}3-8{,}7)} \Rightarrow L_1(8{,}4) = 0{,}75$$

$$L_2(x) = \frac{(x-8{,}1)(x-8{,}3)(x-8{,}7)}{(8{,}6-8{,}1)(8{,}6-8{,}3)(8{,}6-8{,}7)} \Rightarrow L_2(8{,}4) = 0{,}6$$

$$L_3(x) = \frac{(x-8{,}1)(x-8{,}3)(x-8{,}6)}{(8{,}7-8{,}1)(8{,}7-8{,}3)(8{,}7-8{,}6)} \Rightarrow L_3(8{,}4) = -0{,}25$$

$$\begin{aligned} P_3(8{,}4) &= f(8{,}1)L_0(8{,}4) + f(8{,}3)L_1(8{,}4) + f(8{,}6)L_2(8{,}4) + f(8{,}7)L_3(8{,}4) \\ &= -1{,}69441 + 13{,}17369 + 11{,}10309 - 4.7052275 \\ &= \boxed{17{,}87714} \end{aligned}$$

✓

DEGRÉ 2 : $f(8{,}1) = 16{,}94410$, $f(8{,}3) = 17{,}56492$, $f(8{,}6) = 18{,}50515$

$$L_0(x) = \frac{(x-8{,}3)(x-8{,}6)}{(8{,}1-8{,}3)(8{,}1-8{,}6)} \Rightarrow L_0(8{,}4) = -0{,}2$$

$$L_1(x) = \frac{(x-8{,}1)(x-8{,}6)}{(8{,}3-8{,}1)(8{,}3-8{,}6)} \Rightarrow L_1(8{,}4) = 1$$

$$L_2(x) = \frac{(x-8{,}1)(x-8{,}3)}{(8{,}6-8{,}1)(8{,}6-8{,}3)} \Rightarrow L_2(8{,}4) = 0{,}2$$

$$\begin{aligned} P_2(8{,}4) &= f(8{,}1)L_0(8{,}4) + f(8{,}3)L_1(8{,}4) + f(8{,}6)L_2(8{,}4) \\ &= -3{,}38882 + 17{,}56492 + 3{,}70103 \\ &= \boxed{17{,}87713} \end{aligned}$$

✓

<u>DEGRE 1</u> : $f(8{,}1) = 16{,}94410$, $f(8{,}3) = 17{,}56492$

$$L_0(x) = \frac{(x-8{,}3)}{(8{,}1-8{,}3)} \Rightarrow L_0(8{,}4) = -0{,}5$$

$$L_1(x) = \frac{(x-8{,}1)}{(8{,}3-8{,}1)} \Rightarrow L_1(8{,}4) = 1.5$$

$$p_1(8{,}4) = f(8{,}1)L_0(8{,}4) + f(8{,}3)L_1(8{,}4)$$
$$= -8{,}47205 + 26{,}34738$$
$$= \boxed{17{,}87533}$$

Il est mieux de prendre 8.3 et 8.6 p.c.q.

$8{,}3 < 8.4 < 8.6$

Avec 8.3 et 8.6 $p_1(8{,}4) = 17{,}87833$

4.8 $(-1;2)$, $(0;0)$, $(1{,}5;-1)$, $(2;4)$

→ Polynôme de Newton aux différences divisées - Degré 3

Table de différence divisée.

x	$f(x)$	$1^{ère}$ D.D	2^{e} D.D.	3^{e} D.D.
-1	$\boxed{2}$			
		$\boxed{-2}$		
0	0		$(4/3) \div 2{,}5 = \boxed{0{,}53}$	
		$-2/3$		$(4{,}8) \div 3 = \boxed{1.6}$
1.5	-1		$(32/3) \div 2 = 5{,}33$	
		10		
2	4			

$$p_3(x) = 2 + (-2)(x-(-1)) + (0{,}53)(x-(-1))(x-0) + (1.6)(x-(-1))(x-0)(x-1{,}5)$$
$$= \cancel{2} - 2x - \cancel{2} + 0.53x^2 + 0{,}53x + 1.6x^3 + 1{,}6x^2 - 2{,}4x^2 - 2{,}4x$$
$$= 1{,}6x^3 - 0{,}27x^2 - 3{,}87x$$

Polynôme de Newton Divisé : $\boxed{p_3(x) = x(1.6x^2 - 0{,}27x - 3{,}87)}$

D4.7