Nom / Name : SOLUTIONS

No d'ét. / Stud. No.:

Test mi-session 2
Durée: 90 min
Place: MRT 205
17 mars 2010
17h30–19h00

Prof.: Rémi Vaillancourt

MAT 2784 B

Midterm 2
Time: 90 min
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17 March 2010
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Instructions:

(a) *À livre fermé. Tout type de calculatrices permis.*
Closed book. Any type of calculators is allowed.
(b) *Répondre sur le questionnaire et dans les boîtes.*
Answer on the question sheets and fill boxes.
(c) *Les 6 questions sont d'égale valeur.*
All 6 questions have the same value.
(d) *Donner le détail de vos calculs.*
Show all computation.
(e) *Un formulaire sera distribué.*
Formulae will be distributed.

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
Total	/60

Vecteurs propres et vecteurs propres généralisés

Soit une matrice A qui admet une valeur propre λ de multiplicité r et un seul vecteur propre associé $\boldsymbol{u}_1$. On construit les vecteurs propres généralisés $\{\boldsymbol{u}_2, \dots, \boldsymbol{u}_r\}$ solutions des systèmes

$$(A-\lambda I)\boldsymbol{u}_2 = \boldsymbol{u}_1, \quad (A-\lambda I)\boldsymbol{u}_3 = \boldsymbol{u}_2, \quad \dots, \quad (A-\lambda I)\boldsymbol{u}_r = \boldsymbol{u}_{r-1}.$$

Le vecteur propre $\boldsymbol{u}_1$ et les vecteurs propres généralisés engendrent les solutions linéairement indépendantes de $\boldsymbol{y}' = A\boldsymbol{y}$:

$$\boldsymbol{y}_1(x) = e^{\lambda x}\boldsymbol{u}_1, \quad \boldsymbol{y}_2(x) = e^{\lambda x}(x\boldsymbol{u}_1 + \boldsymbol{u}_2), \quad \boldsymbol{y}_3(x) = e^{\lambda x}\left(\frac{x^2}{2}\boldsymbol{u}_1 + x\boldsymbol{u}_2 + \boldsymbol{u}_3\right), \quad \dots.$$

Qu. 1. *Résoudre.* / Solve.

$$y'' + y = 2\cos x, \quad y(0) = 1, \quad y'(0) = 0.$$

Pour $Ly=0$ posons $y=e^{\lambda x}$ $y'=\lambda e^{\lambda x}$ $y''=\lambda^2 e^{\lambda x}$

$\therefore \quad \lambda^2 e^{\lambda x} + e^{\lambda x} = 0$

$\lambda^2 + 1 = 0 \qquad \lambda_{1,2} = \pm i$

$y_h = C_1 \cos x + C_2 \sin x$

$y_p = A\cos x + B\sin x \Rightarrow$ on multiplie par x, comme $y_p = y_h \ldots$

$y_p = Ax\cos x + Bx\sin x$

$y_p' = A\cos x - Ax\sin x + B\sin x + Bx\cos x$

$y_p'' = -2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x$

$y_p'' + y_p = 2\cos x$

$(-2A\sin x - \cancel{Ax\cos x} + 2B\cos x - \cancel{Bx\sin x}) + (\cancel{Ax\cos x} + \cancel{Bx\sin x}) = 2\cos x$

$-2A\sin x + 2B\cos x = 2\cos x$

$-2A = 0 \quad \therefore A = 0$

$2B = 2 \quad \therefore B = 1$

$y_p = x\sin x$

$y_g(x) = y_h + y_p = C_1\cos x + C_2 \sin x + x\sin x \Leftarrow$ Solution Générale

$y_g(0) = \boxed{C_1 = 1}$

$y_g'(x) = -C_1 \sin x + C_2\cos x + \sin x + x\cos x$

$y_g'(0) = \boxed{C_2 = 0}$

$\therefore$ Solution unique:

$$\boxed{y_g(x) = \cos x + x\sin x}$$

Qu. 1. *Résoudre.* / Solve.

$$y'' + y = 2\cos x, \quad y(0) = 1, \quad y'(0) = 0.$$

*Solution :

L'équation homogène est : $y'' + y = 0$

$\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i \rightarrow$ racines complexes / $\alpha = 0,\ \beta = 1$

Donc : $y_1(x) = e^0 \cos(x) = \cos(x)$, $y_2(x) = e^0 \sin(x) = \sin(x)$

$\boxed{y_h(x) = C_1 \cos x + C_2 \sin x} \rightarrow$ solution homogène.

Trouve $y_p(x)$: Variations de paramètres.

$y_p(x) = U_1(x) \cos x + U_2(x) \sin x$

on a : $\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2\cos x \end{bmatrix} \Rightarrow \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} 0 \\ 2\cos x \end{bmatrix}$

$\Rightarrow u_1'(x) = -2\sin x \cos x \Rightarrow \boxed{u_1(x) = \cos^2 x}$

$u_2'(x) = 2\cos^2 x \Rightarrow u_2(x) = x + \frac{1}{2}\sin 2x$ mon erreur au tableau

Donc : $y_p(x) = \cos^3 x + x\sin x + \frac{1}{2}\sin x \sin 2x$ . .

$y(x) = y_h(x) + y_p(x) = C_1 \cos x + C_2 \sin x + \cos^3 x + x \sin x + \frac{1}{2}\sin x \sin 2x$

$\Rightarrow y'(x) = -C_1 \sin x + C_2 \cos x + 3\sin x \cos^2 x + \sin x + x\cos x + \frac{1}{2}\cos x \sin 2x + \frac{2}{2}\sin x \cos 2x$.

$y(0) = 1 \Rightarrow C_1 + 1 = 1 \Rightarrow \boxed{C_1 = 0}$

$y'(0) = 0 \Rightarrow \boxed{C_2 = 0}$

La solution uniques est

$$\boxed{y(x) = \cos^3 x + x\sin x + \frac{1}{2}\sin x \sin 2x}$$

Remarque : sol. unique $y(x) = y_p(x)$ ici !!

Qu. 1. *Résoudre.* / Solve.

$$y'' + y = 2\cos x, \quad y(0) = 1, \quad y'(0) = 0.$$

PAR LAPLACE

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = 2\,\frac{s}{s^2+1}$$

$$(s^2+1)\,Y(s) = s + 2\,\frac{s}{s^2+1}$$

$$Y(s) = \frac{s}{s^2+1} + 2\,\frac{s}{(s^2+1)^2}$$

$$y(t) = \cos t + t \sin t$$

PUISQUE

$$t \sin t \xrightarrow{\mathcal{L}} -\left(\frac{1}{s^2+1}\right)'$$

$$= -\,\frac{-2s}{(s^2+1)^2}$$

$$= 2\,\frac{s}{(s^2+1)^2}$$

Qu. 2. *Trouver la solution générale.* / Find the general solution.

$$y'' - 2y' + y = \frac{e^x}{x}.$$

* Trouvons y_h.

Posons $y = e^{\lambda x} \Rightarrow y' = \lambda e^x,\ y'' = \lambda^2 e^x$

$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda_{1,2} = \dfrac{2 \pm \sqrt{4 - 4(1)(1)}}{2}$

$\lambda_{1,2} = 1$

$\boxed{y_h = C_1 e^x + C_2 x e^x}$

* Trouvons y_p Par Variation des paramètres

$y_p = A(x) e^x + B(x) x e^x$

$$\begin{bmatrix} A'(x) \\ B'(x) \end{bmatrix} \cdot \begin{bmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^x}{x} \end{bmatrix}$$

$$\begin{bmatrix} A'(x) \\ B'(x) \end{bmatrix} = \frac{1}{e^{2x} + x e^{2x} - x e^{2x}} \begin{bmatrix} e^x + x e^x & -x e^x \\ -e^x & e^x \end{bmatrix} \begin{bmatrix} 0 \\ \frac{e^x}{x} \end{bmatrix}$$

$$= \frac{1}{e^{2x}} \begin{bmatrix} e^x + x e^x & -x e^x \\ -e^x & e^x \end{bmatrix} \begin{bmatrix} 0 \\ \frac{e^x}{x} \end{bmatrix}$$

$\frac{1+x}{e^x}$

$$= \begin{bmatrix} \frac{(1+x)}{e^x} & -\frac{x}{e^x} \\ -\frac{1}{e^x} & \frac{1}{e^x} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{e^x}{x} \end{bmatrix}$$

$$\begin{bmatrix} A'(x) \\ B'(x) \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{x} \end{bmatrix}$$

$A(x) = -x$

$B(x) = \ln|x|$

D'où $y_p = -x e^x + x \ln(x) e^x$

La solution générale est alors

$$\boxed{y(x) = C_1 e^x + C_2 x e^x - x e^x + x \ln(x) e^x}$$

✓

Qu. 3. *Résoudre.* / Solve.

$$\boldsymbol{y}' = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} \boldsymbol{y}, \qquad \boldsymbol{y}_0 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

$$\text{dét}(A - \lambda I) = \begin{vmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = (-3-\lambda)(-1-\lambda) + 2$$

$$= \lambda^2 + 4\lambda + 5 = 0 \qquad \lambda_{1,2} = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$$

$\lambda_1 = -2 + i$

$$(A - \lambda_1 I)\vec{u} = \begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} u_1 = 2 \\ u_2 = 1+i \end{matrix} \quad \vec{u} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$$

$$-(1+i)\,u_1 + 2\,u_2 = 0$$

$$w_1 = e^{-2x} \begin{bmatrix} 2 \\ 1+i \end{bmatrix} e^{ix} = e^{-2x}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)(\cos x + i \sin x)$$

$$= e^{-2x}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cos x - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin x \right) + i e^{2x} \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \sin x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos x \right)$$

$$y_1(x) = \operatorname{Re} w_1(x) = e^{-2x}\left[\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cos x - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin x \right]$$

$$y_2(x) = \operatorname{Im} w_1(x) = e^{-2x}\left[\begin{bmatrix} 2 \\ 1 \end{bmatrix} \sin x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos x \right]$$

Sol. gén.

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x)$$

$$y(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$2c_1 = 1 \Rightarrow c_1 = 1/2$$

$$c_1 + c_2 = -2 \Rightarrow c_2 = -2 - 1/2 = -5/2$$

Sol. (unique)

$$y(x) = \frac{1}{2} y_1(x) - \frac{5}{2} y_2(x)$$

Qu. 4. *Résoudre par Laplace.* / Solve by Laplace transform.

$$\underbrace{y'' - 5y' + 6y}_{f(t)} = \underbrace{\begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}}_{g(t)}, \quad y(0) = 0, \quad y'(0) = 1.$$

1° $F(s) = s^2 Y(s) - s\cancel{y(0)}^{0} - y'(0)^{1} - 5sY(s) + 5\cancel{y(0)}^{0} + 6Y(s) = (s^2 - 5s + 6)Y(s) - 1 = (s-2)(s-3)Y(s) - 1$

2° $g(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \geqslant 1 \end{cases} = t - t\,u(t-1) = t - (t-1+1)u(t-1) = t - u(t-1)(t-1) - u(t-1)(1)$

$(s-2)(s-3) = s^2 - 5s + 6$

$\cdot s = s^3 - 5s^2 + 6s$

$G(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$

3° $(s-2)(s-3)Y(s) - 1 = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \Rightarrow Y(s) = \frac{1}{s^2(s-2)(s-3)} - \frac{e^{-s}}{s^2(s-2)(s-3)} - \frac{e^{-s}}{s(s-2)(s-3)} + \frac{1}{(s-2)(s-3)}$

$\frac{1}{s^2(s-2)(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{D}{s-3} = A(s^3 - 5s^2 + 6s) + B(s^2 - 5s + 6) + C(s^3 - 3s^2) + D(s^3 - 2s)$ — s^2

$= (A + C + D)s^3 + (-5A + B - 3C)s^2 + (6A - 5B - 2D)s + 6B = 1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ -5 & 1 & -3 & -2 & 0 \\ 6 & -5 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 6 & 0 & 0 & 1 \\ 0 & 1 & 2 & 5 & 0 \\ 0 & -5 & -6 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 6 & 0 & 0 & 1 \\ 0 & 0 & 12 & 30 & -1 \\ 0 & 0 & 4 & 17 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 6 & 0 & 0 & 1 \\ 0 & 0 & 12 & 30 & -1 \\ 0 & 0 & 0 & 21 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 22/21 \\ 0 & 1 & 0 & 0 & 1/6 \\ 0 & 0 & 1 & 0 & -71/42 \\ 0 & 0 & 0 & 1 & 9/14 \end{array}\right]$$

X

$A = \cancel{\frac{13}{84}}$ $5/36$

$B = \frac{1}{6}$ ✓

$C = \cancel{\frac{-17}{84}}$ $-1/4$

$D = \cancel{\frac{1}{21}}$ $1/9$

$\frac{1}{s(s-2)(s-3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s-3} = A(s^2 - 5s + 6) + B(s^2 - 3s) + C(s^2 - 2s) = (-5A - 3B - 2C)s + (A + B + C)s^2 + 6A = 1$

$$\left[\begin{array}{ccc|c} 6 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ -5 & -3 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/6 \\ 0 & 1 & 1 & -1/6 \\ 0 & -3 & -2 & 5/6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/6 \\ 0 & 1 & 1 & -1/6 \\ 0 & 0 & 1 & 2/6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/6 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 2/6 \end{array}\right]$$

$A = 1/6$ ✓

$B = -1/2$ ✓

$C = 2/6$ ✓

$\frac{1}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3} = As + Bs - 2B - 3A = (A+B)s - 2B - 3A = 1 \Rightarrow A = -B$ et $-2B - 3A = -2B + 3B = \boxed{B = 1}$ ✓ $\boxed{A = -1}$ ✓

$Y(s) = \left[\frac{13}{84}\frac{1}{s} + \frac{1}{6}\frac{1}{s^2} - \frac{17}{84}\frac{1}{(s-2)} + \frac{1}{21}\frac{1}{s-3}\right](1 - e^{-s}) - e^{-s}\left[\frac{1}{6}\frac{1}{s} - \frac{1}{2}\frac{1}{s-2} + \frac{1}{3}\frac{1}{s-3}\right] + \frac{1}{s-2} - \frac{1}{s-3}$ (⊖ ⊕)

X

$y(t) = \frac{13}{84} + \frac{t}{6} - \frac{17e^{2t}}{84} + \frac{e^{3t}}{21} - u(t-1)\left[\frac{13}{84} + \frac{(t-1)}{6} - \frac{17e^{2(t-1)}}{84} + \frac{e^{3(t-1)}}{21}\right] - u(t-1)\left[\frac{1}{6} - \frac{e^{2(t-1)}}{2} + \frac{e^{3(t-1)}}{3}\right] + e^{2t} - e^{3t}$ (⊖ ⊕)

Rép : $y(t) = \frac{13}{84} + \frac{t}{6} + \frac{67e^{2t}}{84} - \frac{20e^{3t}}{21} - u(t-1)\left[\frac{27}{84} + \frac{(t-1)}{6} - \frac{59e^{2(t-1)}}{84} + \frac{8e^{3(t-1)}}{21}\right]$

Qu. 5. *Compléter le tableau de différences divisées.* / Complete the divided difference table.

i	x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	3.1	21.0			
			11.3333		
1	2.8	17.6		4.3725	
			$\boxed{2.5882}$		−1.4004
2	1.1	13.2		$\boxed{2.1318}$	
			$\boxed{6.6388}$		$\boxed{-4.7101}$
3	4.7	37.1		$\boxed{-10.5855}$	
			$\boxed{-39.9375}$		
4	5.5	5.15			

Construire le polynôme de degré 3 qui interpole les données aux 4 points de $x_0 = 3.1$ *à* $x_3 = 4.7$.

Construct the cubic polynomial that interpolates the data at the four points $x_0 = 3.1$ to $x_3 = 4.7$

$$P(x) = f_0 + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2) f[x_0, x_1, x_2, x_3]$$

$$= 21.0 + (x-3.1)(11.3333) + (x-3.1)(x-2.8)(4.3725) + (x-3.1)(x-2.8)(x-1.1)(-1.4004)$$

$$= 21.0 + 11.3333x - 108.9130 + 4.3725\left[x^2 - 2.8x - 3.1x + 8.68\right]$$
$$+ (-1.4004)\left[x^3 - 1.1x^2 - 2.8x^2 + 3.08x - 3.1x^2 + 3.41x + 8.68x - 9.548\right]$$

$$= 21.0 + 11.3333x - 108.9130 + 4.3725x^2 - 25.7977x + 37.9533$$
$$- 1.4004x^3 + 9.8028x^2 - 21.2440x + 13.3710$$

$$\boxed{P(x) = -1.4004x^3 + 14.1753x - 47.0417x - 36.5887}$$

pas nécessaire

Qu. 6. *Déterminer h et n pour approcher l'intégrale à 10^{-5} près :*
Determine h and n to approximate the integral to an error smaller then 10^{-5}: $= \varepsilon$

$$\int_0^2 \frac{1}{x+4}\,dx \qquad b=2, \quad a=0$$

par la méthode des points milieux / by the composite midpoint rule:

$$\int_a^b f(x)\,dx = h[f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)] + \frac{(b-a)h^2}{24}f''(\xi), \qquad a<\xi<b.$$

$f(x) = \frac{1}{x+4}$ si $u = x+4 \Rightarrow du = dx$ $\int \frac{du}{u} = \int du\,u^{-1} = -u^{-2}$

$f'(x) = -\frac{1}{(x+4)^2}$ dans $\frac{Mh^2(b-a)}{24} \leq 10^{-5}$

$f''(x) = \frac{2}{(x+4)^3}$ $M = \left|\max_{0\leq x\leq 2} f''(x)\right|$ $f''(2) = \frac{2}{(2+4)^3} = 0{,}009\overline{259}$

$\boxed{f''(0) = \frac{2}{(0+4)^3} = 0{,}03125}$

$\therefore M = 0{,}03125$

$$\frac{(0{,}03125)h^2(2-0)}{24} \leq 10^{-5}$$

$$\frac{h^2}{384} \leq 10^{-5} \Rightarrow h^2 \leq 0{,}00384 \Rightarrow h \leq 0{,}061967733$$

$n \cdot h = (b-a)$ $n = \frac{(2-0)}{0{,}061967733} = 32{,}275$

On arrondit n à la hausse $\boxed{\therefore n = 33}$

et $h = \frac{(b-a)}{n} = \frac{(2-0)}{33} = \frac{2}{33}$ $\boxed{\therefore h \cong 0{,}06061}$

✓