Nom / Name : SOLUTIONS

No d'ét. / Stud. No.:

Test mi-session 1
Durée: 90 min
Place: MRT 205
10 février 2010
17h30–19h00

Prof.: Rémi Vaillancourt

MAT 2784 B

Midterm 1
Time: 90 min
Place: MRT 205
10 February 2010
17:30–19:00

Instructions:

(a) *À livre fermé. Tout type de calculatrices permis.*
Closed book. Any type of calculators is allowed.
(b) *Répondre sur le questionnaire et dans les boîtes.*
Answer on the question sheets and fill boxes.
(c) *Les 6 questions sont d'égale valeur.*
All 6 questions have the same value.
(d) *Donner le détail de vos calculs.*
Show all computation.
(e) *Un formulaire sera distribué.*
Formulae will be distributed.

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
Total	/60

L'équation différentielle homogène du 1er ordre, / The first-order homogeneous ODE,

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

admet le facteur d'intégration / admits the integrating factor

$$\mu(x) = e^{\int f(x)\,dx} \qquad \text{si/if} \qquad \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x),$$

ou / or

$$\mu(y) = e^{-\int g(y)\,dy} \qquad \text{si/if} \qquad \frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(y).$$

[redacted]

Qu. 1. *Trouver un facteur d'intégration, rendre l'équation différentielle exacte et résoudre le problème à valeur initiale.*

Find an integration factor, make the differential equation exact and solve the initial value problem.

$(x^2y - x^3)$

$$(1 - x^2y)\,dx + x^2(y - x)\,dy = 0, \qquad y(1) = 2.$$

$M_y = -x^2$ $\quad N_x = 2xy - 3x^2$ $\quad$ pas exact

$$\frac{M_y - N_x}{N} = \frac{-x^2 - 2xy + 3x^2}{x^2(y-x)} = \frac{-2xy + 2x^2}{x^2(y-x)} \; \frac{-2\cancel{x}(\cancel{y - x})}{x^{\cancel{2}}(\cancel{y-x})} = \frac{-2}{x}$$

$$u(x) = e^{\int f(x)\,dx} = x^{-2} = \frac{1}{x^2}$$

$$\left(\frac{1}{x^2} - y\right)dx + (y - x)\,dy = 0$$

$$u(x,y) = \int N\,dy + T(x) = \int (y - x)\,dy + T(x) = \frac{y^2}{2} - xy + T(x)$$

$u_x = -y + T'(x)$ $\qquad T'(x) = \frac{1}{x^2}$

$$T(x) = \int \frac{1}{x^2}\cdot dx = -x^{-1} = \frac{-1}{x}$$

Sol gén : $\frac{y^2}{2} - xy - \frac{1}{x} = C$

$$\frac{2^2}{2} - 2 - 1 = C$$

$$c = -1$$

Sol unique : $\frac{y^2}{2} - xy - \frac{1}{x} = -1$

Qu. 2. *Résoudre l'équation linéaire à valeur initiale.*

Solve the linear equation with given initial value.

$$\frac{xy' + 6y = 3x + 2}{x}, \qquad y(1) = 2.$$

$$= y' + \frac{6}{x} y = \frac{3x+2}{x} = 3 + \frac{2}{x}$$

$$F(x) = \frac{6}{x} \Rightarrow u(x) = e^{\int \frac{6}{x} dx} = x^6$$

$$\int (x^6 \cdot y)' = \left(3 + \frac{2}{x}\right) x^6 = \int \left(3x^6 + 2x^5\right) dx$$

$$x^6 y = \frac{3x^7}{7} + \frac{2x^6}{6} + C$$

sol gén: $y(x) = \frac{3}{7}x + \frac{1}{3} + \frac{C}{x^6}$

$$y(1) = \frac{3}{7}(1) + \frac{1}{3} + \frac{C}{1} = 2$$

$$C = \frac{26}{21}$$

$$y(x) = \frac{3}{7}x + \frac{26}{21x^6} + \frac{1}{3}$$ ✓ solution unique.

Qu. 3. ***Résoudre le problème aux valeurs initiales.***

Solve the initial value problem.

$$y'' - 2y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

$y = e^{\lambda x}$

$y' = \lambda e^{\lambda x}$

$y'' = \lambda^2 e^{\lambda x}$

$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + 2e^{\lambda x} = 0$

$e^{\lambda x} \neq 0$, alors

$\lambda^2 - 2\lambda + 2 = 0$

$\lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = 1 \pm \dfrac{\sqrt{-4}}{2} = 1 \pm i$

$y_1(x) = C_1 e^{\alpha x} \cos \beta x = C_1 e^{x} \cos x$

$y_2(x) = C_2 e^{\alpha x} \sin \beta x = C_2 e^{x} \sin x$

$y(x) = C_1 e^{x} \cos x + C_2 e^{x} \sin x$

$y'(x) = -C_1 e^{x} \sin x + C_2 e^{x} \cos x$

$y(0) = C_1 e^{0} \cos 0 + C_2 e^{0} \sin 0 = C_1 = 1$

$y'(0) = -C_1 e^{0} \sin 0 + C_2 e^{0} \cos 0 = C_2 = 2$

Solution unique: $y(x) = e^{x} \cos x + 2e^{x} \sin x$

✓

Qu. 4. *Résoudre le problème aux valeurs initiales.*

Solve the initial value problem.

équation différentielle homogène

$$x^2y'' + 5xy' + 3y = 0, \quad x > 0, \quad y(1) = 1, \quad y'(1) = -5.$$

Posons $y = x^m$ $\quad y' = mx^{m-1}$ $\quad y'' = (m-1)mx^{m-2}$

$$x^2(m-1)mx^{m-2} + 5x(mx^{m-1}) + 3x^m = 0$$

$$\cancel{x^m}(m^2-m) + 5m\cancel{x^m} + 3\cancel{x^m} = 0$$

$$m^2+4m+3=0 \implies (m+1)(m+3) \qquad m_1=-1, \quad m_2=-3$$

$$m_1 \neq m_2 \checkmark$$

$$y(x) = C_1x^{m_1} + C_2x^{m_2} = C_1x^{-1} + C_2x^{-3}$$

Condition initiale:

$$1 = C_1 \cdot 1^{-1} + C_2 \cdot 1^{-3} = C_1 + C_2 \qquad C_1 = 1 - C_2$$

$$y'(x) = -C_1x^{-2} - 3C_2x^{-4}$$

Condition initiale:

$$-5 = -\underbrace{(1-C_2)}_{C_1 = 1-C_2} \cdot 1^{-2} - 3C_2(1)^{-4} = -1 + C_2 - 3C_2$$

$$-4 = -2C_2 \implies C_2 = 2 \quad \text{et} \quad C_1 = 1-2 = -1$$

$\therefore$ Solution unique:

$$y(x) = -x^{-1} + 2x^{-3}$$

Qu. 5. (a) *Itérer 4 fois la récurrence de point fixe à 6 décimales près.*
Iterate 4 times the fixed point recurrence. Use at least 6 decimals.

$$x_{n+1} = g(x_n), \quad x_0 = 2.5, \quad \text{avec / with} \quad g(x) = \frac{6}{x^2} + 2.$$

et calculer l'erreur si : / and compute the error if: $p = g(p) = 2.777663$.

$x_1 = \boxed{2{,}96}$ $x_1 - p = \boxed{0{,}182337}$

$x_2 = \boxed{2{,}684806}$ $x_2 - p = \boxed{-0{,}092857}$

$x_3 = \boxed{2{,}832387}$ $x_3 - p = \boxed{0{,}054724}$

$x_4 = \boxed{2{,}747904}$ $x_4 - p = \boxed{-0{,}029759}$

$x_5 = \boxed{2.794599}$ $x_5 - p = \boxed{0.016936}$ arrondi !

(b) *Calculer* / Compute $g'(x) = -2 \cdot \frac{6}{x^3} = \frac{-12}{x^3}$

$g'(x_5) = \boxed{-0{,}549823} \approx g'(p) = -0.559940.$

$0 < |g'(x)| < 1$

(c) *Quel est l'ordre p de convergence de la méthode ?*
What is the order p of convergence of the method?

$p = \boxed{1}$

Qu. 6. (a) *Soit* / Consider $f(x) = x^3 - 4x^2 + 5x - 2.$
Calculer / Compute

$f(1) = \boxed{0}$

$f'(1) = \boxed{0}$ $f'(x) = 3x^2 - 8x + 5$

$f''(1) = \boxed{-2}$ $f''(x) = 6x - 8$

(b) *Quelle est la multiplicité* m *du zéro* $x = 1$ de $f(x)$?
What is the multiplicity m of the zero $x = 1$ of $f(x)$?

$m = \boxed{2}$

(c) *Itérer deux fois à 6 décimales la méthode newtonienne modifiée avec* m *en (b)* :
Iterate twice to 6 decimals Newton's modified method with m in (b):

$$x_{n+1} = x_n - m\,\frac{f(x_n)}{f'(x_n)}, \qquad x_0 = 1.5.$$

$x_0 = 1.5$

$x_1 = \boxed{0,5}$

$x_2 = \boxed{0,928571}$

(d) *Quelle est l'ordre de convergence* p *de la méthode en (c)* ?
What is the order of convergence p of the method in (c)?

$p = \boxed{* \ 2}$