Nom / Name : SOLUTIONS

No d'ét. / Stud. No.: ____________________

MAT 2784 B

Examen final
Durée: 3h
Place: GYM C
21 avril 2010
19h–22h

Final Exam
Time: 3h
Place: GYM C
21 April 2010
19:00–22:00

Prof.: Rémi Vaillancourt

Instructions:

(a) *À livre fermé. Tout type de calculatrices permis.*
Closed book. Any type of calculators is allowed.

(b) *Répondre sur le questionnaire et dans les boîtes.*
Answer on the question sheets and fill boxes.

(c) *Les 8 questions sont d'égale valeur.*
All 8 questions have the same value.

(d) *Donner le détail de vos calculs.*
Show all computation.

(e) *Il y a deux pages de tables à la fin du questionnaire.*
A two-page table is at the end of the questionnaire.

(f) *Tous les angles sont en RADIANS.*
Tester et ajuster votre calculatrice.
All angles are in RADIAN measures.
Test and adjust your calculators.

$$\sin 1.123456789 = 0.90160112364453$$

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
8	/10
Total	/80

Vecteurs propres multiples et vecteurs propres généralisés
Multiple eigenvalues and generalized eigenvectors

Soit une matrice A qui admet une valeur propre λ de multiplicité r et un *seul* vecteur propre associé $\boldsymbol{u}_1$. On construit les vecteurs propres généralisés $\{\boldsymbol{u}_2, \dots, \boldsymbol{u}_r\}$ solutions des systèmes

$$(A - \lambda I)\boldsymbol{u}_2 = \boldsymbol{u}_1, \quad (A - \lambda I)\boldsymbol{u}_3 = \boldsymbol{u}_2, \quad \dots, \quad (A - \lambda I)\boldsymbol{u}_r = \boldsymbol{u}_{r-1}.$$

Le vecteur propre $\boldsymbol{u}_1$ et les vecteurs propres généralisés engendrent ~~les~~ solutions linéairement indépendantes de $\boldsymbol{y}' = A\boldsymbol{y}$:

$$\boldsymbol{y}_1(x) = e^{\lambda x}\boldsymbol{u}_1, \quad \boldsymbol{y}_2(x) = e^{\lambda x}(x\boldsymbol{u}_1 + \boldsymbol{u}_2), \quad \boldsymbol{y}_3(x) = e^{\lambda x}\left(\frac{x^2}{2}\boldsymbol{u}_1 + x\boldsymbol{u}_2 + \boldsymbol{u}_3\right), \quad \dots.$$

Qu. 1. *Résoudre le problème à valeur initiale.* / Solve the initial value problem.

$$\overset{M}{(x+y^2)}\,dx - \overset{N}{2xy}\,dy = 0, \qquad y(1) = 2.$$

1° $M_y = 2y \neq N_x = -2y$ Pas exacte

2° $f(x) = \dfrac{M_y - N_x}{N} = \dfrac{2y+2y}{-2xy} = \dfrac{-4y}{2xy} = \dfrac{-2}{x}$

3° $\mu(x) = e^{\int f(x)\,dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\ln x} = e^{\ln \frac{1}{x^2}} = \boxed{\dfrac{1}{x^2}}$ facteur d'intégration

$\Rightarrow \frac{1}{x^2}(x+y^2)dx - \frac{2xy}{x^2}\,dy = \overset{M}{\left(\frac{1}{x} + \frac{y^2}{x^2}\right)}dx - \overset{N}{\frac{2y}{x}}\,dy = 0 = du$

4° $M_y = \dfrac{2y}{x^2} = N_x = \dfrac{2y}{x^2}$ ✓ exacte

5° $u(x,y) = \int u_y\,dy + T(x) = -2\int \frac{y}{x}\,dy + T(x) = \frac{-y^2}{x} + T(x)$

6° $u_x = \cancel{\frac{y^2}{x^2}} + T'(x) = \frac{1}{x} + \cancel{\frac{y^2}{x^2}}$ $\Rightarrow T'(x) = \frac{1}{x}$ et $T(x) = \int \frac{1}{x}dx = \ln|x|$

Sol générale: $u(x,y) = \frac{-y^2}{x} + \ln|x| = C$

7° $y(1) = 2$: $\frac{-(2)^2}{1} + \ln|1| = \boxed{-4 = C}$

Rép: $u(x,y) = \frac{-y^2}{x} + \ln|x| = -4$ ✓

Qu. 2. *Résoudre le problème à valeur initiale.* / Solve the initial value problem.

$$x^2y'' + 5xy' + 4y = 0, \qquad x > 0,\ y(1) = 2,\ y'(1) = -4.$$

1° Posons $y = x^m$: $x^2 \cdot x^{m-2}(m-1)m + 5x \cdot x^{m-1} \cdot m + 4x^m = 0$

$m(m-1)x^m + 5 \cdot m x^m + 4x^m = \cancel{x^m}(m(m-1) + 5m + 4) = 0 \quad \Rightarrow m^2 - m + 5m + 4 = m^2 + 4m + 4 = (m+2)^2 = 0$

On a $m_{1,2} = -2$ d'où $y_1 = x^{-2}$ et $y_2 = x^{-2} \ln x$

Sol. générale : $y(x) = C_1 x^{-2} + C_2 x^{-2} \ln x$

2° $y(1) = 2$ et $y'(1) = -4$

$\Rightarrow 2 = C_1 (1)^{-2} + \cancel{C_2 (1)^{-2} \ln 1} = \boxed{C_1 = 2}$

$\dfrac{C_2 x^{-2}}{x} = C_2 x^{-3}$

$\Rightarrow y'(x) = -2C_1 x^{-3} - 2C_2 x^{-3} \ln x + C_2 x^{-3}$

$-4 = -2C_1 (1)^{-3} - \cancel{2C_2 (1)^{-3} \ln(1)} + C_2 (1)^{-3}$

$-4 = -2C_1 + C_2 \quad \Rightarrow -4 = -2(2) + C_2 \quad \Rightarrow C_2 = -4 + 4 = 0$

Rép : $y(x) = \dfrac{2}{x^2}$ ✓

√ **Qu. 3.** *Trouver la solution générale.* / Find the general solution.

$$y'' - 2y' + y = \frac{e^x}{x}.$$ $r(x) \rightsquigarrow$ dérivées infinies

1° $L_{y_h} = y'' - 2y' + y = 0$ Posons $y = e^{\lambda x}$: $\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0$

$\cancel{e^{\lambda x}}(\lambda^2 - 2\lambda + 1) = 0 \Rightarrow \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$

On a $\lambda_1 = 1$ et $\lambda_2 = 1$ d'où $y_1 = e^x$ et $y_2 = xe^x$

$y_h = c_1 e^x + c_2 x e^x$

2° $y_p = c_1(x) e^x + c_2(x) x e^x$

$$\Rightarrow \begin{bmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{bmatrix} \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^x}{x} \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} e^x & xe^x & 0 \\ e^x & xe^x + e^x & \frac{e^x}{x} \end{array}\right] \sim \left[\begin{array}{cc|c} e^x & xe^x & 0 \\ 0 & e^x & \frac{e^x}{x} \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & x & 0 \\ 0 & 1 & \frac{1}{x} \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{x} \end{array}\right]$$

$\rightarrow c_1'(x) = -1$ et $c_1(x) = -x$

$\rightarrow c_2'(x) = \frac{1}{x}$ et $c_2(x) = \ln|x|$

d'où $y_p = -xe^x + xe^x \ln|x|$

Sol générale : $y(x) = c_1 e^x + c_2 x e^x - xe^x + xe^x \ln|x|$

Qu. 4. *Résoudre* / Solve :

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

n.b. : *Voir page titre* / See front page.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda)(2-\lambda) = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 2$$

$$(A - 2I)u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0 \quad \Rightarrow u_2 = 0 \quad u_3 = 0$$ et posons $u_1 = 1$

$$\Rightarrow \vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v_2 = 1, \quad v_3 = 0, \quad v_1 = 0 \quad \Rightarrow \vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow w_3 = 1, \quad w_2 = w_1 = 0 \quad \Rightarrow \vec{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$y_g = c_1 \vec{u} e^{2x} + c_2 e^{2x}(\vec{u}x + \vec{v}) + c_3 e^{2x}\left(\vec{u}\frac{x^2}{2} + \vec{v}x + \vec{w}\right)$$

$$y(0) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow \begin{aligned} c_1 &= 1 \\ c_2 &= -1 \\ c_3 &= 2 \end{aligned}$$

$$y_p = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2x} - e^{2x}\left(x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) + 2e^{2x}\left(\frac{x^2}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$

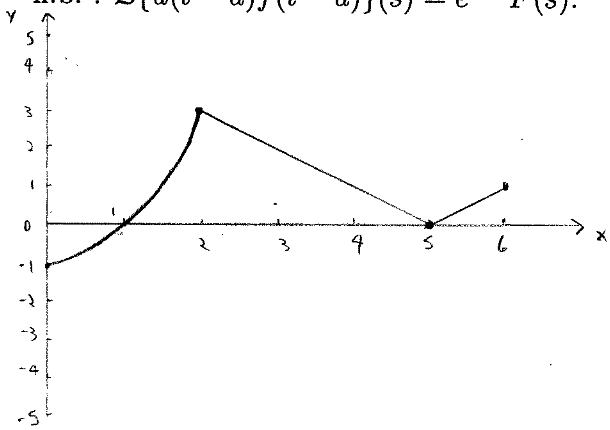
Qu. 5. *Tracer $h(t)$ sur $[0,6]$ (2/10), développer $h(t)$ suivant la fonction $u(t-a)$ d'Heaviside (3/10) et trouver la transformée de Laplace de $h(t)$ (5/10).*

Sketch the graph of $h(t)$ on $[0,6]$ (2/10), expand $h(t)$ in terms of Heaviside's function $u(t-a)$ (3/10) and find the Laplace transform of $h(t)$ (5/10).

$$h(t) = \begin{cases} -1+t^2, & 0 \le t < 2, \\ 5-t, & 2 \le t < 5, \\ t-5, & 5 \le t. \end{cases}$$

n.b. : $\mathcal{L}\{u(t-a)f(t-a)\}(s) = e^{-as}F(s)$.

1)



a)

2) $h(t) = (t^2-1) - u(t-2)(t^2-1) + u(t-2)(5-t) - u(t-5)(5-t) + u(t-5)(t-5)$

$h(t) = t^2-1 - u(t-2)t^2 + u(t-2) - u(t-2)(t-5) + 2u(t-5)(t-5)$

3) 1° $-u(t-2)t^2 = -u(t-2)((t-2)+2)^2 = -u(t-2)((t-2)^2 + \cancel{2}4(t-2) + 4) = -u(t-2)(t-2)^2 - \cancel{2}4u(t-2)(t-2) - 4u(t-2)$

2° $-u(t-2)(t-5) = -u(t-2)(t-2-5+2) = -u(t-2)(t-2-3) = -u(t-2)(t-2) + 3u(t-2)$

d'où $h(t) = t^2-1 - u(t-2)(t-2)^2 - 3u(t-2)(t-2) \cancel{-4u(t-2)} \cancel{+4u(t-2)} + 2u(t-5)(t-5)$

$h(t) = t^2-1 - u(t-2)(t-2)^2 - \cancel{3}4u(t-2)(t-2) + 2u(t-5)(t-5)$

$\mathcal{L}(h(t))(s) = \frac{2}{s^3} - \frac{1}{s} - \frac{2e^{-2s}}{s^3} - \frac{\cancel{3}4e^{-2s}}{s^2} + \frac{2e^{-5s}}{s^2}$

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✓**Qu. 6.** ***Résoudre par transformation de Laplace.*** / Solve by Laplace transforms.

$$y'' + 4y' + 8y = \delta(t-3), \qquad y(0) = 1,\ y'(0) = 2.$$

n.b. : $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$.

$$\frac{-4 \pm \sqrt{16-32}}{2} = \frac{-4 \pm \sqrt{-16}}{2}$$
$$= -2 \pm 2i$$

1° $s^2 Y(s) - sY(0) - Y'(0) + 4sY(s) - 4Y(0) + 8Y(s) = (s^2+4s+8)Y(s) - s - 2 - 4 = (s^2+4s+8)Y(s) - s -$

$= e^{-3s} \Rightarrow Y(s) = \dfrac{e^{-3s} + s + 6}{(s^2+4s+8)} = \dfrac{e^{-3s} + s + 6}{(s^2+4s+4)+4} = e^{-3s} + s + 6 \left[\dfrac{1}{2} \cdot \dfrac{2}{(s+2)^2+2^2}\right]$

$= \dfrac{1}{2}\left[\dfrac{2e^{-3s}}{(s+2)^2+2^2} + 2 \cdot \underbrace{\dfrac{s}{(s+2)^2+2^2}} + 6 \cdot \dfrac{2}{(s+2)^2+2^2}\right]$

$\boxed{s = (s+2) - 2}$ d'où $= 2\left[\dfrac{s+2}{(s+2)^2+2^2} - \dfrac{2}{(s+2)^2+2^2}\right]$

2° $Y(t) = \dfrac{1}{2}\left[u(t-3)\, e^{-2(t-3)} \sin(-2(t-3)) + 2e^{-2t}\cos(-2t) - 2e^{-2t}\sin(-2t) + 6e^{-2t}\sin(-2t)\right]$

$$\boxed{Y(t) = 0{,}5\, u(t-3)\, e^{-2(t-3)} \sin(-2(t-3)) + e^{-2t}\cos(-2t) + 2e^{-2t}\sin(-2t)}$$

✓

√ **Qu. 7.** *Soit* / Let $f(x) = e^x \ln x$. $f'(x) = e^x \ln x + \frac{e^x}{x}$ et $f'(2) = 8{,}816231451$

Par Richardson, extrapoler $f'(2)$ *obtenue par différence centrée avec* $h = 0.2, 0.1, 0.05$.
By Richardson, extrapolate $f'(2)$ obtained by central difference with $h = 0.2, 0.1, 0.05$.

$$N_1(0.2) = N(0.2) = \frac{1}{0.4}[f(2.2) - f(1.8)] = \boxed{8{,}899840535}$$

$$N_1(0.1) = N(0.1) = \frac{1}{0.2}[f(2.1) - f(1.9)] = \boxed{8.837095456}$$

$$N_1(0.05) = N(0.05) = \frac{1}{0.1}[f(2.05) - f(1.95)] = \boxed{8{,}821445064}$$

Ensuite / Next

$$N_2(0.2) = N_1(0.1) + \frac{N_1(0.1) - N_1(0.2)}{3} = \boxed{8{,}816180043}$$

$$N_2(0.1) = N_1(0.05) + \frac{N_1(0.05) - N_1(0.1)}{3} = \boxed{8.816228267}$$

Enfin / Finally

$$N_3(0.2) = N_2(0.1) + \frac{N_2(0.1) - N_2(0.2)}{15} = \boxed{8.816231456}$$

Calculer $f'(2)$ *exactement et vérifier que* $N_3(0.2)$ *est exacte à 6 décimales près* :
Compute $f'(2)$ exactly and verify that $N_3(0.2)$ is exact to 6 decimals:

$$A := \frac{d}{dx}(e^x \ln x)\Big|_{x=2} = \boxed{8{,}816231451}$$

$$\Big|A - N_3(0.2)\Big| = \boxed{-5 \times 10^{-9}} \leq 10^{-6}. \quad \checkmark$$

√ **Qu. 8.**

MATLAB ode23 *pour* / for $y' = f(x,y)$, $y(x_0) = y_0$:

$$\begin{aligned}
k_1 &= hf(\overset{1}{x_n}, \overset{0}{y_n}),\\
k_2 &= hf(x_n \overset{1{,}05}{+} (1/2)h, y_n \overset{0{,}05}{+} (1/2)k_1),\\
k_3 &= hf(x_n \overset{1{,}075}{+} (3/4)h, y_n \overset{0{,}0714375}{+} (3/4)k_2),\\
k_4 &= hf(x_n \overset{1{,}1}{+} h, y_n + (2/9)k_1 \overset{0{,}095627}{+} (1/3)k_2 + (4/9)k_3),
\end{aligned}$$

$$y_{n+1} = y_n + (2/9)k_1 + (1/3)k_2 + (4/9)k_3,$$

avec estimation de l'erreur locale / with local error estimate:

$$E_{n+1} = -\frac{5}{72}k_1 + \frac{1}{12}k_2 + \frac{1}{9}k_3 - \frac{1}{8}k_4.$$

Compléter les 5 petites cases pour l'équadif et les 8 ***grandes cases à 6 décimales***:
Fill-in the 5 the small boxes for the ode and the ~~7~~ 8 **long boxes to six decimals:**

$$y' = y^2 - 2y + x, \qquad y(1) = 0, \qquad h = 0.1.$$

Pour / For $n = 0$:

$f(x,y) =$ $\boxed{y^2-2y+x}$ $x_0 =$ $\boxed{1}$ $y_0 =$ $\boxed{0}$ $h =$ $\boxed{0{,}1}$

$k_1 =$ $\boxed{0{,}100000}$

$k_2 =$ $\boxed{0{,}095250}$

$k_3 =$ $\boxed{0{,}093723}$

$k_4 =$ $\boxed{0{,}091789}$

$y_1 =$ $\boxed{0{,}095627}$

$E_1 =$ $\boxed{-0{,}000067}$

Pour / For $n = 1$:

$x_1 =$ $\boxed{1{,}1}$ $y_1 =$ $\boxed{0{,}095627}$

$k_1 =$ $\boxed{0{,}091789}$

✓