2010. 03. 03
MAT 2784B

Devoir #5

4.4. $A = \begin{bmatrix} -1 & 1 & 4 \\ -2 & 2 & 4 \\ -1 & 0 & 4 \end{bmatrix}$ $y' = Ay$

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 1 & 4 \\ -2 & 2-\lambda & 4 \\ -1 & 0 & 4-\lambda \end{vmatrix} = -1 \begin{vmatrix} 1 & 4 \\ 2-\lambda & 4 \end{vmatrix} + (4-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ -2 & 2-\lambda \end{vmatrix} = -1(4-8+4\lambda) + (4-\lambda)(\cancel{-2}-2\lambda+\lambda^2+2)$$

$$= (4-4\lambda) + (-4\lambda + 4\lambda^2 + \lambda^2 - \lambda^3) = 4 - 8\lambda + 5\lambda^2 - \lambda^3 = (\lambda-1)(-\lambda^2+4\lambda-4) = (\lambda-1)(\lambda-2)(-\lambda+2)$$

$$= -1\cdot(\lambda-1)(\lambda-2)(\lambda-2) \qquad \lambda_1 = 1 \qquad \lambda_{2,3} = 2$$

$$A - \lambda_1 I = \begin{bmatrix} -2 & 1 & 4 \\ -2 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 1/2 & 2 \\ 0 & 0 & 0 \\ 0 & -1/2 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Il faut écrire u et non μ (mu grec)

$$(A-I)u = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = 0 \quad \text{posons } \mu_1 = 1 \implies e_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$A - \lambda_2 I = A - \lambda_3 I = \begin{bmatrix} -3 & 1 & 4 \\ -2 & 0 & 4 \\ -1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A-2I)v = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \quad \text{posons } v_3 = 1 \implies e_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad \text{et} \quad e_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution générale:

$$C_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} e^{x} + C_2 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{2x} + C_3\left(x \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) e^{2x}$$

Il faut dire que e_3 est solution de
$(A-2I)e_3 = e_2$.

4.8. $y' = Ay + f(x)$ où $A = \begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}$ et $f(x) = \begin{bmatrix}2e^{-x}\\ 3x\end{bmatrix}$ D5.2

$\det(A-\lambda I) = (-2-\lambda)(-2-\lambda) - 1\cdot 1 = 4 + 4\lambda + \lambda^2 - 1 = \lambda^2 + 4\lambda + 3 = (\lambda+1)(\lambda+3)$

$\lambda_1 = -1 \quad \lambda_2 = -3$

$(A+I) = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix} \quad u = \begin{bmatrix}1\\1\end{bmatrix} \qquad (A+3I) = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \quad v = \begin{bmatrix}1\\-1\end{bmatrix}$

$y_h(x) = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{-x} + c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-3x}$

$y_p(x) = \begin{bmatrix}a_1\\a_2\end{bmatrix}xe^{-x} + \begin{bmatrix}b_1\\b_2\end{bmatrix}e^{-x} + \begin{bmatrix}c_1\\c_2\end{bmatrix}x + \begin{bmatrix}d_1\\d_2\end{bmatrix}$

$y_p'(x) = -\begin{bmatrix}a_1\\a_2\end{bmatrix}xe^{-x} + \begin{bmatrix}a_1\\a_2\end{bmatrix}e^{-x} - \begin{bmatrix}b_1\\b_2\end{bmatrix}e^{-x} + \begin{bmatrix}c_1\\c_2\end{bmatrix}$

$y' = Ay + f(x)$

$-\begin{bmatrix}a_1\\a_2\end{bmatrix}xe^{-x} + \begin{bmatrix}a_1\\a_2\end{bmatrix}e^{-x} - \begin{bmatrix}b_1\\b_2\end{bmatrix}e^{-x} + \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}xe^{-x} + \begin{bmatrix}b_1\\b_2\end{bmatrix}e^{-x} + \begin{bmatrix}c_1\\c_2\end{bmatrix}x + \begin{bmatrix}d_1\\d_2\end{bmatrix}\right) + \begin{bmatrix}2e^{-x}\\3x\end{bmatrix}$

$-\begin{bmatrix}a_1\\a_2\end{bmatrix}xe^{-x} + \begin{bmatrix}a_1-b_1\\a_2-b_2\end{bmatrix}e^{-x} + \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}-2a_1+a_2\\a_1-2a_2\end{bmatrix}xe^{-x} + \begin{bmatrix}-2b_1+b_2\\b_1-2b_2\end{bmatrix}e^{-x} + \begin{bmatrix}-2c_1+c_2\\c_1-2c_2\end{bmatrix}x + \begin{bmatrix}-2d_1+d_2\\d_1-2d_2\end{bmatrix} + \begin{bmatrix}2\\0\end{bmatrix}e^{-x} + \begin{bmatrix}0\\3\end{bmatrix}x$

$-a_1 = -2a_1 + a_2 \Rightarrow a_1 = a_2 = a$ posons $a_1 = 1 \Rightarrow a = 1$

$a_1 - b_1 = -2b_1 + b_2 + 2 \Rightarrow b_1 = b_2 + 1$ posons $b_1 = 2 \Rightarrow b_2 = 1$

$a_2 - b_2 = b_1 - 2b_2 + 0 \Rightarrow 1 + b_2 = b_1$ ✓ (vérifié)

$\left.\begin{array}{l}0 = -2c_1 + c_2 + 0\\ 0 = c_1 - 2c_2 + 3\end{array}\right\} \left.\begin{array}{l}2c_1 = c_2\\ -3 + 2c_2 = c_1\end{array}\right\} \left.\begin{array}{l}c_1 = \tfrac12 c_2\\ -3 + 2c_2 = \tfrac12 c_2\end{array}\right\} \begin{array}{l}c_1 = 1\\ c_2 = 2\end{array}$

$\left.\begin{array}{l}c_1 = -2d_1 + d_2\\ c_2 = d_1 - 2d_2\end{array}\right\} \left.\begin{array}{l}1 + 2d_1 = d_2\\ 2 + 2(1+2d_1) = d_1\end{array}\right\} \begin{array}{l}d_1 = -4/3\\ d_2 = -5/3\end{array}$

<u>Solution générale:</u>

$y_g(x) = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{-x} + c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-3x} + \begin{bmatrix}1\\1\end{bmatrix}xe^{-x} + \begin{bmatrix}2\\1\end{bmatrix}e^{-x} + \begin{bmatrix}1\\2\end{bmatrix}x - \frac{1}{3}\begin{bmatrix}4\\5\end{bmatrix}$

4.10 $y' = Ay + f(x)$ $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$, $f(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. D5.3

$\det(A-\lambda I) = (1-\lambda)(-1-\lambda) - \sqrt{3}\cdot\sqrt{3} = (-1+\lambda^2-3) = (\lambda^2-4) = (\lambda-2)(\lambda+2)$ $\lambda_1 = 2$ $\lambda_2 = -2$

$(A-2I) = \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -\sqrt{3} \\ 0 & 0 \end{bmatrix}$ posons $u_2 = 1$ $u = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ $(A+2I) = \begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$ posons $V_1 = 1$ $V = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$

$y_h(x) = C_1 \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} e^{-2x}$ $y_p(x) = \begin{bmatrix} a \\ b \end{bmatrix}$ $y_p'(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Puisque $y' = Ay + f(x)$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a+\sqrt{3}b \\ \sqrt{3}a - b \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\left.\begin{matrix} a+\sqrt{3}b+1 = 0 \\ \sqrt{3}a - b + 0 = 0 \end{matrix}\right\} \left.\begin{matrix} a+\sqrt{3}b+1=0 \\ \sqrt{3}a = b \end{matrix}\right\}$ $a + \sqrt{3}(\sqrt{3}a) + 1 = 0 \Longrightarrow 4a+1 = 0$ $\therefore a = -1/4$

$b = -\sqrt{3}/4$

<u>Solution Générale:</u> $y_g(x) = C_1 \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} e^{2x} + C_2 \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} e^{-2x} - \frac{1}{4} \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$

4.11 $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$ $y(0) = y_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $y' = Ay$

$\det(A-\lambda I) = (5-\lambda)(1-\lambda) - (3\cdot -1) = 5 - 6\lambda + \lambda^2 + 3 = \lambda^2 - 6\lambda + 8 = (\lambda-4)(\lambda-2)$

$\lambda_1 = 4$ $\lambda_2 = 2$

$(A-4I) = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ posons $u_1 = 1$ $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $(A-2I) = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$ Posons $V_2 = 1$ $V = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Solution Générale: $y(x) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4x} + C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2x}$

$y(0) = \begin{bmatrix} C_1 \\ C_1 \end{bmatrix} + \begin{bmatrix} C_2 \\ 3C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\left.\begin{matrix} C_1 + C_2 = 2 \\ C_1 + 3C_2 = -1 \end{matrix}\right\}$ $2 - C_2 = C_1$ $(2-C_2) + 3C_2 = -1$ $-3 = 2C_2$ $C_2 = -3/2$

$\therefore C_1 = 7/2$

<u>Solution unique:</u> $y(x) = 7/2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4x} - 3/2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2x}$

5.5. Trouver l'intervalle de convergence de $\sum_{n=3}^{\infty} \frac{n(n-1)(n-2)}{4^n} x^n$ et de sa 1ère dérivée terme à terme.

D 5.4

1° $a_n = \frac{n(n-1)(n-2)}{4^n}$ $\quad a_{n+1} = \frac{(n+1)(n)(n-1)}{4^{n+1}}$

$$1/R = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)\cancel{(n)}\cancel{(n-1)}}{4^{n+1}} \cdot \frac{4^n}{n(n-1)(n-2)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)}{4(n-2)}\right|$$

$$= \lim_{n\to\infty} \frac{1}{4}\left|\frac{\cancel{n}(1+1/n)}{\cancel{n}(1-2/n}\right| = \lim_{n\to\infty} \frac{1}{4}\left|\frac{(1+1/\cancel{n})}{(1-2/n)}\right| = \frac{1}{4}\frac{(1)}{(1)} = 1/4$$

$1/R = 1/4 \therefore R=4$

intervalle de convergence: $-4<x<4$

2° $F'(x) = \frac{n^2(n-1)(n-2)}{4^n} x^{n-1} = \frac{n^2(n-1)(n-2)x^n}{x \cdot 4^n}$

$a_n = \frac{n^2(n-1)(n-2)}{x\, 4^n}$ $\quad a_{n+1} = \frac{(n+1)^2(n)(n-1)}{x \cdot 4^{n+1}}$

$$1/R' = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2(n)\cancel{(n-1)}}{x \cdot 4^{n+1}} \cdot \frac{x \cdot 4^n}{n^2(n-1)(n-2)}\right| = \lim_{n\to\infty}\left|\frac{(n^2+2n+1)}{4 n(n-2)}\right| = \lim_{n\to\infty}\left|\frac{(n^2+2n+1)}{4(n^2-2n)}\right|$$

$$\lim_{n\to\infty}\frac{1}{4}\left|\frac{\cancel{n^2}(1+2/n+1/n^2)}{\cancel{n^2}(1-2/n)}\right| = \frac{1}{4}\frac{(1)}{(1)} = \frac{1}{4} \quad \therefore R' = 4$$

intervalle de convergence: $-4<x<4$

5.8 Trouver l'intervalle de convergence de $\sum_{n=1}^{\infty} \frac{(4n)!}{(n!)^4} x^n$ et de sa 1ère dérivée terme à terme.

$$1°\ 1/R = \lim_{n\to\infty}\left|\frac{(4n+4)!}{[(n+1)!]^4} \cdot \frac{(n!)^4}{(4n)!}\right| = \lim_{n\to\infty}\left|\left[\frac{(n!)}{(n+1)!}\right]^4 \cdot \frac{(4n+4)!}{(4n)!}\right| = \lim_{n\to\infty}\left|\left(\frac{1}{n+1}\right)^4 \cdot (4n+4)(4n+3)(4n+2)(4n+1)\right|$$

$$= \lim_{n\to\infty}\left|\frac{4^4 n^4 + 640n^3 + 560n^2 + 200n + 24}{n^4 + 4n^3 + 6n^2 + 4n + 1}\right| = \lim_{n\to\infty}\left|\frac{\cancel{n^4}(4^4 + 640/n + 560/n^2 + 200/n^3 + 24/n^4)}{\cancel{n^4}(1 + 4/n + 6/n^2 + 4/n^3 + 1/n^4)}\right|$$

$= 4^4 = 256 \therefore R = 1/256$ intervalle de convergence: $-1/256 < x < 1/256$

2° $F'(x) = \sum_{n=2}^{\infty} \frac{n(4n)!}{(n!)^4} x^{n-1}$ si $m=n-1$ $\quad F'(x) = \sum_{n=1}^{\infty} \frac{(m+1)(4m+4)!}{[(m+1)!]^4} x^m$

$$1/R' = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\left|\frac{(m+2)(4m+8)!}{[(m+2)!]^4} \cdot \frac{[(m+1)!]^4}{(m+1)(4m+4)!}\right| = \lim_{m\to\infty}\left|\left(\frac{(m+1)!}{(m+2)!}\right)^4 \cdot \frac{(m+2)(4m+8)!}{(m+1)(4m+4)!}\right|$$

$$\lim_{m\to\infty}\left|\left(\frac{1}{(m+2)}\right)^4\left(\frac{(m+2)}{(m+1)}\right)(4m+8)(4m+7)(4m+6)(4m+5)\right| = \lim_{m\to\infty}\left|\frac{4^4 m^4 + \cdots}{(m+1)(m+2)^3}\right|$$

En isolant m^4 au dénominateur, la limite de $m\to\infty$ nous laissera seulement 4^4 et en isolant m^4 au numérateur, la limite de $m\to\infty$ nous laissera seulement 1

$1/R' = 4^4 = 256 \therefore R' = 1/256$

intervalle de convergence: $-1/256 < x < 1/256$

9.9. Interpoler les données $(-1,2), (0,0), (1,-1), (2,4)$ par un polynôme de Gregory-Newton de degré 3.

D 5.5

i	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
0	-1	2			
			-2		
1	0	0		1	
			-1		$\boxed{5}$
2	1	-1		$\boxed{6}$	
			$\boxed{5}$		
3	2	$\boxed{4}$			

$r = \left(\frac{x - x_0}{h}\right) = \frac{x-(-1)}{1} = x+1$ où h = intervalle de x_i

$$P_3(r) = \frac{4}{0!} + \frac{r(5)}{1!} + \frac{r(r+1)}{2!}(6) + \frac{r(r+1)(r+2)}{3!}(5)$$

$$P_3(x+1) = 4 + 5x+5 + (x^2+3x+2)(3) + (x^3+6x^2+11x+6)(5/6)$$
$$P_3(x+1) = 5/6\, x^3 + 8x^2 + 139/6\, x + 20$$

10.1 Soit $f'(x_0) = 1/2h[f(x_0+h) - f(x_0-h)] + \frac{h^2}{6} f^{(3)}(\xi)$ (DN.5)
selon le tableau $\{x_n, f(x_n)\}$ de la p. 304 du manuel), $h=0,1$

a) $dfn = f'(1,2) = \frac{1}{0,2}[f(1,3) - f(1,1)] = 5(0,272531793034\,01 - 0,332871083698\,08)$

$= 5(-0,0603392906407) = -0,3016964533\,2035$

b) $f(x) = \cosh(x) - \sinh(x)$ $f'(x) = \sinh(x) - \cosh(x)$

$dfc = f'(1,2) = \sinh(1,2) - \cosh(1,2) = \frac{e^{1,2} - e^{-1,2}}{2} - \frac{e^{1,2}+e^{-1,2}}{2} = -e^{-1,2} = -0,3011942119$

c) $\varepsilon = dfn - dfc = -0,30169645332035 - (-0,3011942119) = 0,00050224142035$

d) $|\varepsilon| \leq \frac{h^2}{6}|f^{(3)}(x)|$ $0,00050224142035 \leq \frac{(0,1)^2}{6}|-e^{-1,1}|$

$0,00050224142035 \leq 0,0005547851395$ ✓