2010.02.24
MAT 2784 B

Devoir #4

D4.1

3.7 $L=y'''-y''-y'+y=0$; $y(0)=0$, $y'(0)=5$, $y''(0)=2$ Équation linéaire homogè[ne] à coefficients consta[nts]

posons $y=e^{\lambda x}$; $y'=\lambda e^{\lambda x}$; $y''=\lambda^2 e^{\lambda x}$; $y'''=\lambda^3 e^{\lambda x}$

$\lambda^3 e^{\lambda x}-\lambda^2 e^{\lambda x}-\lambda e^{\lambda x}+e^{\lambda x}=0 \Longrightarrow \lambda^3-\lambda^2-\lambda+1=0=(\lambda-1)(\lambda^2-1)$

$0=(\lambda-1)(\lambda-1)(\lambda+1)=(\lambda-1)^2(\lambda+1)$ $\lambda_1=1$ $\lambda_2=1$ $\lambda_3=-1$

$y_p(x)=c_1e^x+c_2xe^x+c_3e^{-x}$ $y_p'(x)=c_1e^x+c_2e^x+c_2xe^x-c_3e^{-x}$

$y_p''(x)=c_1e^x+c_2(2e^x+xe^x)+c_3e^{-x}$ $y_p'''(x)=c_1e^x+c_2(3e^x+xe^x)-c_3e^{-x}$

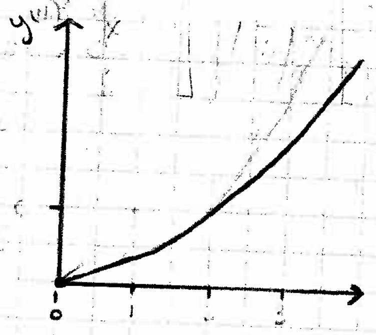
$y(0)=c_1+c_3=0 \iff c_1=-c_3$

$y'(0)=c_1+c_2-c_3=5 \iff -c_3+c_2-c_3=5 \iff c_2=5+2c_3$

$y''(0)=c_1+2c_2+c_3=2 \iff -c_3+2(5+2c_3)+c_3=2 \iff c_3=\frac{2-10}{4}=-2$

$\therefore c_1=2$ et $c_2=1$

$y(x)=2e^x+xe^x-2e^{-x}$



3.11 $y_1(x)=2$, $y_2(x)=\sin^2 x$, $y_3(x)=\cos^2 x$

$$\begin{array}{c} y\to\\ y'\to\\ y''\to\end{array}\begin{vmatrix}2 & \sin^2 x & \cos^2 x\\ 0 & 2\sin x\cdot\cos x & -2\cos x\cdot\sin x\\ 0 & -2\sin^2x+2\cos^2x & -2\cos^2x+2\sin^2x\end{vmatrix} = \begin{vmatrix}2&\frac{1-\cos(2x)}{2}&\frac{1+\cos(2x)}{2}\\0&\sin(2x)&-\sin(2x)\\0&2\cos(2x)&-2\cos(2x)\end{vmatrix}$$

$$l_1+\tfrac14 l_3 = \begin{vmatrix}2&1/2&1/2\\0&\sin(2x)&-\sin(2x)\\0&2\cos(2x)&-2\cos(2x)\end{vmatrix} = 2\sin(2x)\cdot\cos(2x)\begin{vmatrix}2&1/2&1/2\\0&1&-1\\0&1&-1\end{vmatrix}$$

l_3-l_2

$$l_1-\tfrac12 l_2 = \sin(4x)\begin{vmatrix}2&0&1\\0&1&-1\\0&0&0\end{vmatrix}=\sin(4x)\cdot 0=0$$

donc les fonctions sont <u>linéairement dépendantes</u>

identités trigonométriques utilisées:

$\sin(2A)=2\sin(A)\cdot\cos(A)$ $\cos(2A)=\cos^2(A)-\sin^2(A)$

$\sin^2(A)=\frac{1-\cos(2A)}{2}$ $\cos^2(A)=\frac{1+\cos(2A)}{2}$

3.21 (*) $(1+2x)y'' + 4xy' - 4y = 0$, $y_1(x) = e^{-2x}$ $y_2(x) = u(x)y_1(x)$

$y_1'(x) = -2e^{-2x}$

$y_1''(x) = 4e^{-2x}$

$y_2'(x) = u'y_1 + uy_1'$

$y_2''(x) = u''y_1 + u'y_1' + u'y_1' + uy_1'' = u''y_1 + 2u'y_1' + uy_1''$

(*) $(1+2x)(u''y_1 + 2u'y_1' + uy_1'') + 4x(u'y_1 + uy_1') - 4uy = 0$ (D4.2

$u''y_1 + 2u'y_1' + uy_1'' + 2xu''y_1 + 4xu'y_1' + 2xuy_1'' + 4xu'y_1 + 4xuy_1' - 4uy = 0$

$u(y_1'' + 2xy_1'' + 4xy_1' - 4y) + u'(2y_1' + 4xy_1' + 4xy_1) + u''(y_1 + 2xy_1) = 0$

$u(4e^{-2x} + 8xe^{-2x} - 8xe^{-2x} - 4e^{-2x}) + u'(-4e^{-2x} - 8xe^{-2x} + 4xe^{-2x}) + u''(e^{-2x} + 2xe^{-2x}) = 0$

$u'(-4e^{-2x} - 4xe^{-2x}) + u''(e^{-2x} + 2xe^{-2x}) = u'e^{-2x}(-4-4x) + u''e^{-2x}(1+2x) = 0$

$u'(-4-4x) + u''(1+2x) = 0$

si $v = u'$ $v' = u''$ $\Rightarrow$ $v(-4-4x) + v'(1+2x) = 0$

$4v(1+x) = \frac{dv}{dx}(1+2x)$

$C + \int \frac{4(1+x)}{(1+2x)}dx = \int \frac{dv}{v} = C + \int 2dx + \int \frac{2}{(2x+1)}dx = \int \frac{dv}{v}$

$\ln|v| = 2x + \ln|2x+1| + 1$

$v = e^{2x+\ln(2x+1)+1} = e^{2x+1}(2x+1) = u'(x)$

$u(x) = \int u'(x) = k\int e^{2x+1}(2x+1)dx = \int 2xe^{2x+1}dx + \int e^{2x+1}dx$
$= xe^{2x+1}$

$y_2(x) = xe^{2x+1}(e^{-2x}) = ex$ $\Longrightarrow$ peut être simplifié à x (cste va multiplier $y_2(x)$ de toute façon)

$y(x) = C_1y_1(x) + C_2y_2(x) = C_1e^{-2x} + C_2xe^{-2x}$

[solution générale]

3.24 $Ly = y'' - y' - 2y = 2xe^{-x} + x^2$ posons $y = e^{\lambda x}$

$Ly = 0$ $\lambda^2e^{\lambda x} - \lambda e^{\lambda x} - 2e^{\lambda x} = 0$ $\Longrightarrow \lambda^2 - \lambda - 2 = 0$ $\lambda_{1,2} = \frac{1 \pm 3}{2}$

$\therefore y_h(x) = C_1e^{-x} + C_2e^{2x}$

$y_p(x)$: $y_{p_1}(x)$ pour $2xe^{-x} \rightarrow y_{p_1}(x) = (ax+b)e^{-x} = axe^{-x} + be^{-x}$

$y_{p_1}(x) \cdot x \Longrightarrow$ $y_{p_1}(x) = ax^2e^{-x} + bxe^{-x}$

$y'_{p_1}(x) = e^{-x}(2ax - ax^2) + e^{-x}(b - bx)$
$= -e^{-x}(ax^2 + bx) + e^{-x}(2ax + b)$

$y''_{p_1}(x) = (e^{-x})(ax^2 + bx) - e^{-x}(2ax+b) - e^{-x}(2ax+b) + e^{-x}(2a)$

$y''_{p_1} - y'_{p_1} - 2y_{p_1} = e^{-x}(2a - 4ax - 2b + ax^2 + bx + ax^2 + bx - 2ax - b - 2ax^2 - 2bx)$
$= e^{-x}(2a - 6ax - 3b) = 2xe^{-x}$

$\Rightarrow 2x = 2a - 6ax - 3b$ $2a - 3b = 0$ $b = \frac{2a}{3}$

$\Rightarrow 2x = -6ax$ $a = -2/6 = -1/3 \Rightarrow b = -2/9$

ur x^2 $y_{p_2}(x) = ax^2 + bx + c$

$y'_{p_2}(x) = 2ax + b$

$y''_{p_2}(x) = 2a$

$y''_{p_2} - y'_{p_2} - 2y_{p_2} = 2a - 2ax - b - 2ax^2 - 2bx - 2c = x^2$

$\Rightarrow -2ax^2 = x^2$ $a = -1/2$

$\Rightarrow -2ax - 2bx = 0$ $-2a = 2b$ $b = 1/2$

$\Rightarrow 2a - b - 2c = 0$ $2(-1/2) - 1/2 - 2c = 0$ $c = -3/4$

Soln générale: $y(x) = y_h(x) + y_p(x) = C_1e^{-x} + C_2e^{2x} - (1/3x^2 + 2/9x)e^{-x} - 1/2x^2 + 1/2x - 3/4$

3.31 $y'' + y = 1/\cos x = \sec x$ $\boxed{D4.3}$

$Ly = 0 \quad \lambda^2 + 1 = 0 \quad \lambda_{1,2} = 0 \pm i \cdot 1 \Rightarrow y_h(x) = C_1 \cos x + C_2 \sin x$

Variation des paramètres: $\alpha = 0$, $\beta = 1$

$y_p = C_1(x)\cos x + C_2(x)\sin x$

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\cos x \end{bmatrix}$$

① $C_1' \cos x + C_2' \sin x = 0 \Rightarrow C_1' = -\dfrac{C_2' \sin x}{\cos x} = -C_2' \tan x$

② $-C_1' \sin x + C_2' \cos x = 1/\cos x \Rightarrow \dfrac{C_2' \sin x}{\cos x} \cdot \sin x + C_2' \cos x = 1/\cos x$

$\Rightarrow C_2' \left(\dfrac{\sin^2 x + \cos^2 x}{\cos(x)} \right) = C_2' \left(1/\cos x \right) = 1/\cos x \iff C_2' = 1$, $C_1' = -\tan x$

$C_1 = \int C_1' = -\int \tan x \, dx = -\ln|\sec x|$

$C_2 = \int C_2' = \int dx = x$

$y_g = C_1 \cos x + C_2 \sin x - \ln|\sec x| \cdot \cos x + x \sin x$

3.39 $2x^2 y'' + xy' - 3y = 2x^{-3}$, $y(1) = 0$, $y'(1) = 3$ $\quad y'' + \dfrac{y'}{2x} - \dfrac{3y}{2x^2} = x^{-5}$

$Ly = 0$

posons $y = x^m$, $y' = mx^{m-1}$, $y'' = (m-1)mx^{m-2} = (m^2 - m)x^{m-2}$

$2x^m(m^2 - m) + mx^m - 3x^m = 0 \Rightarrow 2m^2 - m - 3 = 0 \quad m_{1,2} = \dfrac{1 \pm 5}{4}$

$m_1 = 1{,}5 = 3/2 \quad m_2 = -1$

$y_h(x) = C_1 x^{3/2} + C_2 x^{-1}$

$y_p(x) = C_1(x) x^{3/2} + C_2(x) x^{-1}$

$$\begin{bmatrix} x^{3/2} & x^{-1} \\ 3/2\, x^{1/2} & -x^{-2} \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ x^{-5} \end{bmatrix}$$

$C_1' x^{1,5} + C_2' x^{-1} = 0 \iff C_1' = -C_2' x^{-5/2}$

$3/2\, C_1' x^{1/2} - C_2' x^{-2} = x^{-5}$

$3/2 \left(-C_2' x^{-2,5}\right) x^{1/2} C_2' x^{-2} = C_2' \left(-3/2\, x^{-2} - x^{-2}\right) = C_2' \left(-5/2\, x^{-2}\right) = x^{-5}$

$C_2' = -\dfrac{2}{5} x^{-3}$

$C_2 = \int -2/5\, x^{-3} dx = 1/5\, x^{-2}$

$C_1' = \left(\dfrac{2}{5} x^{-3}\right)\left(x^{-5/2}\right) = \dfrac{2}{5} x^{-11/2} \quad C_1 = \int 2/5\, x^{-11/2} dx = \dfrac{-4}{45} x^{-9/2}$

Solution générale: $y(x) = Ax^{3/2} + Bx^{-1} - \dfrac{4}{45} x^{-3} + 1/5\, x^{-3} = Ax^{3/2} + Bx^{-1} + \dfrac{1}{9} x^{-3}$

$y'(x) = 3/2\, Ax^{1/2} - Bx^{-2} - \dfrac{1}{3} x^{-4}$

Avec conditions initiales: $y(1) = A + B + \dfrac{1}{9} = 0 \quad A = 1/9 - B$

$y'(1) = 3/2(1/9 - B) - B - \dfrac{1}{3} = 3 \quad B = -19/15$

$\therefore A = 62/45$

Solution unique: $y(x) = 62/45\, x^{3/2} - 19/15\, x^{-1} + 1/9\, x^{-3}$

$= \dfrac{58}{45} x^{3/2} - \dfrac{7}{5} x^{-1} + \dfrac{1}{9} x^{-3}$

9.3 $f(x) = e^{2x}\cos 3x$ en $x_0 = 0,\ x_1 = 0{,}3 \quad x_2 = 0{,}6$

$$p_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x)$$

$$L_0(x) = \frac{(x-0{,}3)(x-0{,}6)}{(0-0{,}3)(0-0{,}6)} = \frac{(x-0{,}3)(x-0{,}6)}{0{,}18}$$

$$L_1(x) = \frac{(x-0)(x-0{,}6)}{(0{,}3-0)(0{,}3-0{,}6)} = \frac{-x(x-0{,}6)}{0{,}09}$$

$$L_2(x) = \frac{(x-0)(x-0{,}3)}{(0{,}6-0)(0{,}6-0{,}3)} = \frac{x(x-0{,}3)}{0{,}18}$$

$$p_2(x) = \frac{x^2-0{,}9x+0{,}18}{0{,}18} - \frac{(x^2-0{,}6x)}{0{,}09}e^{0{,}6}\cos(0{,}9) + \frac{(x^2-0{,}3x)}{0{,}18}e^{1{,}2}\cos(1)$$

$$= \left(\frac{1}{18} - \frac{e^{0.06}}{0{,}09}\cos(0{,}9) + \frac{e^{1{,}2}}{0{,}18}\cos(1{,}8)\right)x^2 + \left(\frac{-0{,}9}{0{,}18} + \frac{0{,}6}{0{,}09}e^{0{,}6}\cos 0{,}9 \cdot \frac{0{,}3}{0{,}18}e^{1{,}2}\cos(1{,}8)\right)x + \frac{0{,}18}{0{,}18}$$

$$= -11{,}22x^2 + 3{,}81x + 1$$

9.5

i	x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}$
0	3,2	22,0			
			8,400		
1	2,7	17,0		2,856	
			2,118		-0,528
2	1.0	14.2		2,012	
			6,342		-4,273
3	4.8	38.3		-10,381	
			-41,413		
4	5.6	5.17			

$$f_3(x) = 22,01 + (x-3,2)(8,400) + (x-3,2)(x-2,7)(2,856) + (x-3,2)(x-2,7)(x-1,0)(-0,523)$$

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