Mercredi, 3 Février 2010

Devoir #3 MAT 2784B.

Question 2.1

$$y'' - 3y' + 2y = 0$$

$$Ly = y'' - 3y' + 2y$$

$$y = e^{\lambda x}$$
$$y' = \lambda e^{\lambda x}$$
$$y'' = \lambda^2 e^{\lambda x}$$

$$Ly = \lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2e^{\lambda x} = 0$$
$$e^{\lambda x}(\lambda^2 - 3\lambda + 2) = 0$$

$$\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2)$$

$$\Rightarrow \lambda_1 = 1.$$
$$\lambda_2 = 2$$

$$y_1(x) = e^x$$

$$y_2(x) = e^{2x}$$

D'où $\dfrac{y_1(x)}{y_2(x)} = \dfrac{e^x}{e^{2x}} = e^{x-2x} = e^{-x} \neq \text{cste}.$

D'où la sol. générale. $\boxed{y(x) = C_1 e^x + C_2 e^{2x}}$

Question 2.6:

$$y'' - 4y' + 3y = 0 \quad ; y(0) = 6 \quad ; y'(0) = 0$$

$$Ly = \lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 3e^{\lambda x} = 0$$

$$e^{\lambda x}(\lambda^2 - 4x + 3) = 0$$

$$\lambda^2 - 4x + 3 = 0$$
$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda_1 = 1 \quad \Rightarrow y_1(x) = e^x$$

$$\lambda_2 = 3 \quad \Rightarrow y_2(x) = e^{3x}$$

Alors: $\dfrac{y_1(x)}{y_2(x)} = e^{-2x} \neq \text{const.}$

D'où la solution générale est: $y = C_1 e^x + C_2 e^{3x}$

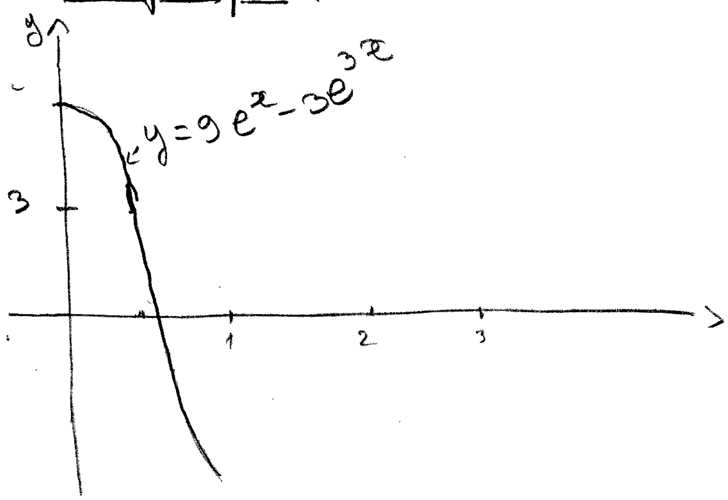
$y(x) = C_1 e^x + C_2 e^{3x} \Rightarrow y(0) = C_1 + C_2 = 6 \Rightarrow C_1 = 6 - C_2$

$y'(x) = C_1 e^x + 3C_2 e^{3x} \quad \Rightarrow y'(0) = C_1 + 3C_2 = 0 \Rightarrow 6 - C_2 + 3C_2 = 0$

$$\Rightarrow \boxed{\begin{array}{l} C_2 = -3 \\ C_1 = 9 \end{array}}$$

La solution Unique est: $\boxed{y = 9e^x - 3e^{3x}}$

Graphique :



Question 2.10

$$y'' + 16y = 0 \quad ; \quad y(0) = 0 \,,\ y'(0) = 1.$$

$$y = e^{\lambda x}$$

$$e^{\lambda x}(\lambda^2 + 16) = 0$$

$$\lambda_{1,2} = \frac{-0 \pm \sqrt{0 - 4 \times 16}}{2} = \frac{\pm\sqrt{-64}}{2}$$

$$= \pm \frac{8i}{2} = \pm 4i \implies \boxed{w = 4}$$

$$y'(t) = C_1 \cos wt + C_2 \sin wt$$

$$y(t) = C_1 \cos 4t + C_2 \sin 4t$$

$$P = \frac{2\pi}{w} = \frac{\pi}{2}$$

On sait que $y(0) = 0$.

$$\implies C_1 + 0 = 0 \implies \boxed{C_1 = 0}$$

$$y'(t) = -C_1 \sin(4t) \times 4 + C_2 \cos(4t) \times 4$$

$y'(0) = 1$

$\Rightarrow 0 + C_2 \times 4 = 1 \Rightarrow \boxed{C_2 = \frac{1}{4}}$

Las solution unique est : $\boxed{y(t) = \frac{1}{4} \sin 4t}$

L'amplitude $A = \sqrt{C_1^2 + C_2^2} = \frac{1}{4}$

et la période $P = \frac{2\pi}{\omega} = \frac{\pi}{2}$.

D'où $\boxed{A = \frac{1}{4} \text{ et } P = \frac{\pi}{2}}$

Question 2.11

$y'' + 2y' + y = 0 \,;\, y(0) = 1 \,;\, y'(0) = 1$

Posons $y = e^{\lambda x}$

$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + e^{\lambda x} = 0$

$e^{\lambda x}(\lambda^2 + 2\lambda + 1) = 0$

$(\lambda + 1)^2 = 0$

$\lambda_{1,2} = \frac{-2 \pm \sqrt{(-1)(4(1) - (2)^2)}}{2} = -1 = -\alpha$.

La solution générale est :

$y(t) = c_1 e^{-\alpha t} + c_2 t e^{-\alpha t}$

$y(t) = c_1 e^{-1t} + c_2 t e^{-t}$

On sait que: $y(0) = 1$ et $y'(0) = 1$.

Alors:

$$y(0) = C_1 + C_2(0) = 1.$$

$$\Rightarrow \underline{C_1 = 1.}$$

- $y'(0) = -C_1 e^{-t} + C_2 \cdot e^{-t} + (-C_2 t e^{-t}) = 1$

$$\Rightarrow -C_1 e^{-t} + C_2 e^{-t} - C_2 t e^{-t} = 1$$

$$\Rightarrow -e^{-t} + C_2 e^{-t} - C_2 t e^{-t} = 1$$

$$\Rightarrow -1 + C_2 - 0 = 1.$$

D'où $\underline{C_2 = 2.}$

La solution unique est:

$$\boxed{y(t) = e^{-t} + 2t\, e^{-t}}$$

~~max~~ Trouvons le max

$$y'(t) = -e^{-t} + 2e^{-t} - 2te^{-t} = 0$$

$$e^{-t}(-1 + 2 - 2t) = 0$$

$$1 - 2t = 0$$

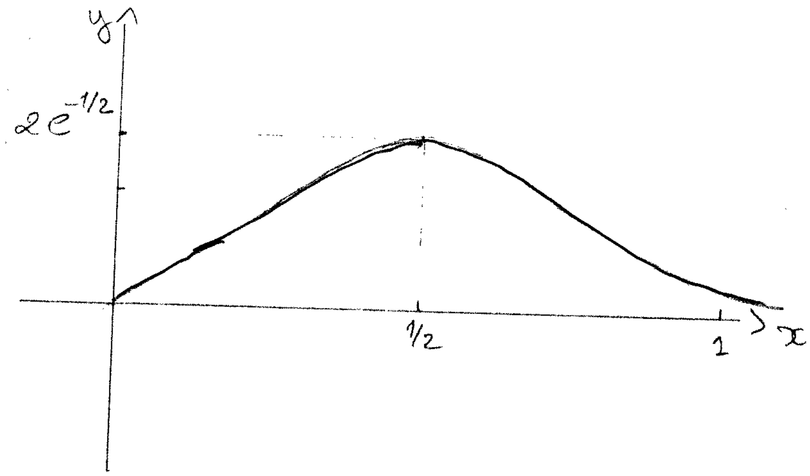
$$2t = 1.$$

$$\Rightarrow t = \frac{1}{2}$$

$$\max = y(1/2)$$

$$y(1/2) = e^{-1/2} + 2(1/2)\, e^{-1/2} = e^{-1/2} + e^{-1/2}$$

$$y(1/2) = 2e^{-1/2} \simeq 1{,}21306$$



<u>Question 2.15:</u> Equation diff. d'Euler - Cauchy

$$4x^2 y'' + y = 0 \quad (*)$$

Posons $y = x^m$ si $x > 0$

$$\Rightarrow y = x^m$$
$$y' = m x^{m-1}$$
$$y'' = m(m-1)x^{m-2}$$

D'où $(*)$ devient :

$$4x^2\left(m(m-1)x^{m-2}\right) + x^m = 0$$

$$4\cancel{x^2 x^{-2}}\, x^m (m^2 - m) + x^m = 0$$

$$4x^m(m^2 - m) + x^m = 0$$

$$x^m(4m^2 - 4m + 1) = 0$$

$$4m^2 - 4m + 1 = 0 \; ; \; x \neq 0$$

$$4\left(m \overset{-}{\cancel{+}} \tfrac{1}{2}\right)^2 = 0$$

$$\left(m \overset{-}{\cancel{+}} \tfrac{1}{2}\right)^2 = 0$$

Les Valeurs propres étants réelles et égales :
$m = m_1 = m_2 = \frac{1}{2}$.

La solution est alors:

$$\boxed{y(x) = c_1 x^{1/2} + c_2 (\ln x) x^{1/2}}$$

x

Question 2.19

(*) $x^2 y'' - xy' + y = 0$; $y(1) = 1$ et $y'(1) = 0$

(*) donne: $m^2 + (-1-1)m + 1 = 0$
$\Rightarrow m^2 - 2m + 1$.

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4(1)(1)}}{2} = 1.$$

$$y(x) = c_1 x^m + c_2 \ln(x)\, x^m$$

$$= c_1 x + c_2 \ln(x)\, x.$$

$y(1) = 1 \Rightarrow \underline{c_1 = 1}$

$y'(x) = c_1 + c_2 + c_2 \ln(x)$

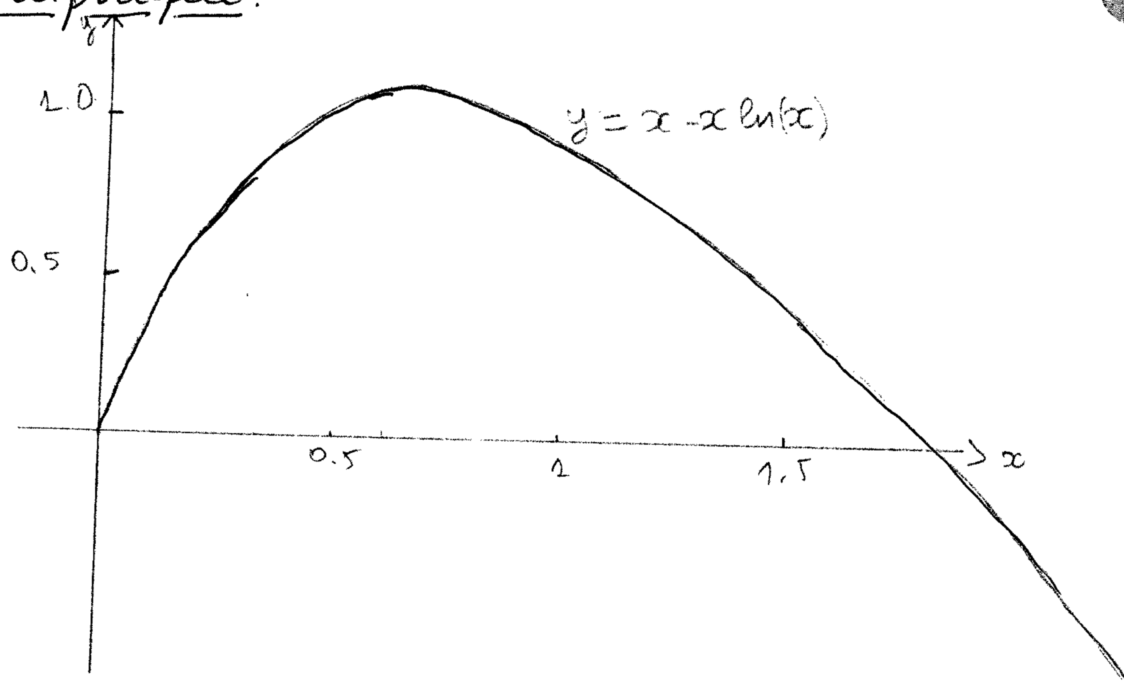
$y'(1) = 0 \Rightarrow c_1 + c_2 = 0$

$\Rightarrow c_2 = -c_1$.

$\Rightarrow \boxed{c_2 = -1}$

D'où $\boxed{y(x) = x - x\ln(x)}$

Graphique:



Question 8.16

$$x_{n+1} = \sqrt{2}\, x_n + 3$$

$x_1 = 4.000$

$x_2 = \sqrt{2x_1 + 3} = 3.31662479$

$\Delta x_1 = x_2 - x_1 = -0.68337520$

$x_3 = \sqrt{2x_2 + 3} = 3.10374767$

$\Delta x_2 = x_3 - x_2 = -0.21287713$

$\Delta^2 x_1 = x_3 - 2x_2 + x_1$

$= 3.10374767 - 2(3.31662479) + 4.000$

$\Delta^2 x_1 = 0.47049808$

$$a_1 = x_1 - \frac{(\Delta x_1)}{\Delta^2 x_1}$$

$$a_1 = 4.000 - \frac{(-0.68337520)^2}{(0.47049808 7)}$$

$$\underline{a_1 = 3.007431323}$$

n	x_n	Δx_n	$\Delta^2 x_n$
1	$x_1 = 4.000$		
		$\Delta x_1 = -0.68337520$	
2	$x_2 = 3.31662479$		$\Delta^2 x_1 = 0.47049808$
		$\Delta x_2 = -0.21287712$	
3	$x_3 = 3.10374767$		

$$a_1 = x_1 - \frac{(\Delta x_1)^2}{\Delta^2 x_1} = \underline{3.007431323}$$

<u>Question 8.20</u>

$$f(x) = x - \tan x \Rightarrow f(0) = 0.$$

$$f'(x) = 1 - \frac{1}{\cos^2 x} \Rightarrow f'(0) = 1 - 1 = 0$$

$$f''(x) = -\frac{2\sin x}{\cos^3 x} \Rightarrow f''(0) = 0$$

$$f'''(x) = -2\left[\frac{\cos^4 x + 3\sin^2 x \cos^2 x}{\cos^6 x}\right] \Rightarrow f'''(0) = -2 \neq 0$$

Newton modifié devient alors:

$$x_{n+1} = x_n - 3\frac{f(x_n)}{f'(x_n)}$$

Car il y a une racine triple et une convergence d'ordre 2.

n	x_{n+1}
0	
1	0.310572
2	0.008100
3	0.0000001
4	—

La racine trouvée est 0.