Nom / Name : SOLUTION

No d'ét. / Stud. No.:

Test mi-session 2
Durée: 75 min
Place: FTX 351
20 novembre 2009
10:05–11:20

Prof.: Rémi Vaillancourt

MAT 2784 A

Midterm 2
Time: 75 min
Place: FTX 351
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Instructions:

(a) *À livre fermé. Tout type de calculatrices permis.*
Closed book. Any type of calculators is allowed.
(b) *Répondre sur le questionnaire.*
Answer on the question sheets.
(c) *Les 5 questions sont d'égale valeur.*
The 5 questions have the same value.
(d) *Donner le détail de vos calculs.*
Show all computation.
(e) *Un formulaire sera distribué.*
Formulae will be distributed.

1	/10
2	/10
3	/10
4	/10
5	/10
Total	/50

Qu. 1. *Trouver la solution générale.* / Find the general solution.

$$y'' + y' = 3x^2.$$

On pose $y = e^{\lambda x} \Rightarrow y' = \lambda e^{\lambda x} \Rightarrow y'' = \lambda^2 e^{\lambda x}$

$\Rightarrow e^{\lambda x}(\lambda^2 + \lambda) = 0$, $e^{\lambda x} \neq 0$

$\lambda(\lambda+1) = 0 \Rightarrow \lambda_1 = 0$, $\lambda_2 = -1$

$y_h(x) = c_1 + c_2 e^{-x}$

Variation des paramètres

$$\begin{bmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 3x^2 \end{bmatrix}.$$

$\Rightarrow c_1' + c_2' e^{-x} = 0$

$-c_2' e^{-x} = 3x^2 \Rightarrow c_2 = -3\int \underbrace{x^2}_{u} \underbrace{e^x\, dx}_{dv}$

$c_1' - (3x^2e^x)e^{-x} = 0$ $\quad = -3[x^2e^x - \int \underbrace{2x}_{u} \underbrace{e^x dx}_{dv}]$

$c_1' - 3x^2 = 0$ $\quad = -3[x^2e^x - 2(xe^x - \int e^x dx)]$

$c_1' = 3x^2$ $\quad = -3[x^2e^x - 2(xe^x - e^x)]$

$c_1 = x^3$ $\quad = -3[x^2e^x - 2xe^x + 2e^x]$

$= -3x^2e^x + 6xe^x - 6e^x$

$= -3e^x(x^2 + 6x - 6)$

$y_P(x) = x^3 + (-3)(x^2+6x-6)e^x ~~e^{-x}~~$

$= x^3 + (-3)(x^2+6x-6).$

$\Rightarrow y_g(x) = y_h(x) + y_P(x)$

$y_g(x) = \textcircled{A} + Be^{-x} + x^3 - 3(x^2+6x\textcircled{-6})$

$A - 6 = \tilde{A}$

Qu. 2. *Trouver la solution générale.* / Find the general solution.

$$y'' + y = \frac{1}{\sin x}.$$

On pose $y = e^{\lambda x} \Rightarrow y' = \lambda e^{\lambda x} \Rightarrow y'' = \lambda^2 e^{\lambda x}$

$\Rightarrow e^{\lambda x}(\lambda^2 + 1) = 0 \quad , e^{\lambda x} \neq 0$

$\lambda^2 = -1$

$\Rightarrow \lambda_{1,2} = \pm i \quad , \alpha = 0 , \beta = 1$

$y_h(x) = A\cos x + B\sin x$

Variation des paramètres.

$A^{-1} = A^T$

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sin x} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sin x} \end{bmatrix}$$

$c_1' = -\sin x \cdot \frac{1}{\sin x} = -1$

$\Rightarrow \boxed{c_1 = -x}$

$c_2' = \cos x \cdot \frac{1}{\sin x} = \cot x$

$\Rightarrow \boxed{c_2 = \ln|\sin x|}$

$\Rightarrow y_p(x) = -x\cos x + \ln|\sin x|\sin x$

$y_g(x) = y_h(x) + y_p(x)$

$$\boxed{y_g(x) = A\cos x + B\sin x - x\cos x + \ln|\sin x|\sin x}$$

✓

Qu. 3. *Trouver la solution générale.* / Find the general solution.

$$y' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} y. \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

1- valeurs propres.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = (-\lambda)(-3-\lambda) - (-2)(1)$$
$$= \lambda^2 + 3\lambda + 2 = 0$$
$$= (\lambda+1)(\lambda+2) = 0$$
$$\Rightarrow \lambda_1 = -1 \quad , \lambda_2 = -2$$

2- vecteurs propres

$\lambda_1 = -1$: $(A+I)\vec{u} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{u} \neq 0$

$\Rightarrow u_1 + u_2 = 0$

On pose $u_1 = 1 \Rightarrow u_2 = -1 \Rightarrow \vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = -2$: $(A+2I)\vec{v} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{v} \neq 0$

$\Rightarrow 2v_1 + v_2 = 0$

On pose $v_1 = 1 \Rightarrow v_2 = -2. \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

3- Sol générale.

$$\vec{y}(x) = c_1 e^{-x}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2x}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

ou $\vec{y}(x) = \begin{bmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

✓

Qu. 4. *Soit l'équation différentielle et sa solution en série :*

Consider the differential equation and its series solution:

$$y'' - 3y' + 2y = 0 \quad \text{et / and} \quad y(x) = \sum_{0}^{\infty} a_m x^m.$$

(a) *Trouver a_2, a_3, a_4 en fonction de a_0 et a_1 et le terme général a_s de la solution en série.*
Find a_2, a_3, a_4 as functions of a_0 and a_1, and the general term a_s of the series solution.
(b) *Quel est le rayon de convergence de la série ?.*
What is the radius of convergence of the series?

On pose: $y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots + a_s x^s$

$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots + (s+1) a_{s+1} x^s$

$y''(x) = 2a_2 + (3\cdot 2) a_3 x + (4\cdot 3) a_4 x^2 + (5\cdot 4) a_5 x^3 + \cdots + (s+2)(s+1) a_{s+2} x^s$ $+ (6\cdot 5) a_6 x^4$

$\Rightarrow$ $2y(x) = 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + 2a_4 x^4 + \cdots + 2a_s x^s$

$-3y'(x) = -3a_1 - 3(2a_2 x) - 3(3a_3 x^2) - 3(4a_4 x^3) + \cdots + (-3)(s+1) a_{s+1} x^s$ $-3(5a_5 x^4)$

$0 = y'' - 3y' + 2y$

$\Rightarrow 0 = (2a_0 - 3a_1 + 2a_2) + x(2a_1 - 6a_2 + 6a_3) + x^2(2a_2 - 9a_3 + 12a_4)$
$+ x^3(2a_3 - 12a_4 + 20a_5) + x^4(2a_4 - 15a_5 + 30a_6) + \cdots$
$+ x^s(2a_s - 3(s+1)a_{s+1} + (s+2)(s+1)a_{s+2})$

x^0: $2a_0 - 3a_1 + 2a_2 = 0 \Rightarrow \boxed{a_2 = \frac{3}{2}a_1 - a_0}$

x^1: $2a_1 - 6a_2 + 6a_3 = 0 \Rightarrow a_3 = a_2 - \frac{1}{3}a_1 = \frac{9}{6}a_1 - \frac{2}{6}a_1 - a_0$

$\boxed{a_3 = \frac{7}{6}a_1 - a_0}$

x^2: $2a_2 - 9a_3 + 12a_4 = 0 \Rightarrow a_4 = \frac{1}{12}(9a_3 - 2a_2)$

$= \frac{3}{4}a_3 - \frac{1}{6}a_2 = \frac{3}{4}\left(\frac{7}{6}a_1 - a_0\right) - \frac{1}{6}\left(\frac{3}{2}a_1 - a_0\right)$

$= \frac{21}{24}a_1 - \frac{3}{4}a_0 - \frac{3}{12}a_1 + \frac{1}{6}a_0$

$= \frac{21}{24}a_1 - \frac{6}{24}a_1 - \frac{9}{12}a_0 + \frac{2}{12}a_0$

$\boxed{a_4 = \frac{15}{24}a_1 - \frac{7}{12}a_0} = \frac{5}{8}a_1 - \frac{7}{12}a_0$ ✓

x^s: $2a_s - 3(s+1)a_{s+1} + (s+2)(s+1)a_{s+2} = 0$

$\Rightarrow \boxed{a_{s+2} = \frac{3(s+1)a_{s+1} - 2a_s}{(s+2)(s+1)}}$

(b) $\Longrightarrow$ verso

↑ Th 5.2 $\Rightarrow R = +\infty$

$a_{s+1} = \frac{2a_s + (s+2)(s+1)a_{s+2}}{3(s+1)}$

Qu. 5. *Soit le terme de l'erreur de la méthode des points milieux pour* $\int_a^b f(x)\,dx$:

The truncation error of the composite midpoint rule for $\int_a^b f(x)\,dx$ is:

$$\frac{(b-a)h^2}{24}f''(\xi).$$

Déterminer les valeurs de h *et* n *pour approcher l'intégrale suivante à* 10^{-4} *près par la méthode des points milieux* :

Determiner the values of h and n to approximate the following integral to 10^{-4} by the composite midpoint rule :

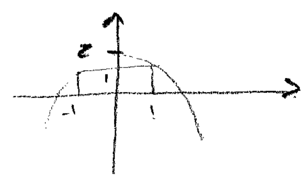
$$\int_{-1}^{1}\left[x^2 - \frac{1}{12}x^4\right]\,dx.$$

$f(x) = x^2 - \frac{1}{12}x^4$

$f'(x) = 2x - \frac{1}{3}x^3$

$f''(x) = 2 - x^2.$

$\max_{-1\le x\le 1} |f''(x)| = f(0) = 2$



$\frac{(b-a)h^2}{24}(2) < 10^{-4}$

$\frac{(1-(-1))(2)h^2}{24} < 10^{-4} \Rightarrow h^2 < 6\cdot 10^{-4}$

$h < 0.024495$

$\Rightarrow \frac{1\times 2}{0.024495} = 2\times 40.8 < 2\times 41 = n.$

$\Rightarrow n = 41\times 2$

$h = \frac{1}{41}$ ✓

(−1)

$nh = b - a = 2$

$n = \frac{2}{h} \Rightarrow n = 82$

$h = \frac{2}{82} = \frac{1}{41}$