**Examen final**
Durée: 3h
Place: GYM F
15 décembre 2009
14h–17h

Prof.: Rémi Vaillancourt

# MAT 2784 A

**Final Exam**
Time: 3h
Place: GYM F
15 December 2009
14:00–17:00

SOLUTIONS

*Nom* / Name: ______________________

*Prénom* / First Name: ______________________

*No d'ét.* / Student ID: ______________________

| | |
|---|---|
| 1 | /10 |
| 2 | /10 |
| 3 | /10 |
| 4 | /10 |
| 5 | /10 |
| 6 | /10 |
| 7 | /10 |
| 8 | /10 |
| Total | /80 |

**Instructions:**

(a) *À livre fermé. Tout type de calculatrices autorisé.*
Closed book. All types of calculators are allowed.

(b) *Répondre sur le questionnaire. Réponses numériques dans les boîtes.*
Answer on the question sheets. Fill-in boxes with numerical answers.

(c) *Les 8 questions sont d'égale valeur.*
All 8 questions have the same value.

(d) *Donner le détail de vos calculs.* / Show all computation.

(e) *Il y a deux pages de tables à la fin du questionnaire.*
A two-page table is at the end of the questionnaire.

(f) *Tous les angles sont en RADIANS. Tester et ajuster votre calculatrice.*
All angles are in RADIAN measures. Test and adjust your calculators.

$$\sin 1.123456789 = 0.90160112364453$$

**Qu. 1.** *Résoudre le problème à valeur initiale.* / Solve the initial value problem.

$$(2x^2y - 2y + 5)\,dx + (2x^3 + 2x)\,dy = 0, \qquad y(1) = 0.$$

$M_y = 2x^2 - 2 \neq N_x = 6x^2 + 2$

Pas exacte

$f(x) = \frac{M_y - N_x}{N} = \frac{(2x^2-2)-(6x^2+2)}{2x^3+2x} = \frac{-4x^2-4}{2x^3+2x} = \frac{-4(x^2+1)}{2x(x^2+1)} = \frac{-4}{2x}$

$= \frac{-2}{x}$

$\mu(x) = e^{\int f(x)dx}$

$= e^{\int \frac{-2}{x}dx}$

$= e^{\ln x^{-2}}$

$= x^{-2}$

$(2y - \frac{2y}{x^2} + \frac{5}{x^2})dx + (2x + \frac{2}{x})dy = 0$

$M_y = 2 - \frac{2}{x^2} = N_x = 2 - \frac{2}{x^2}$

$u(x,y) = \int (2x + \frac{2}{x})dy + T(x)$, $x$ fixé

$= 2xy + \frac{2y}{x} + T(x)$

$u_x'(x,y) = 2y - \frac{2y}{x^2} + T'(x)$

$u_x'(x,y) = M$

$2y - \frac{2y}{x^2} + T'(x) = 2y - \frac{2y}{x^2} + \frac{5}{x^2}$

$T'(x) = \frac{5}{x^2}$

✓ $T(x) = \frac{-5}{x}$

Solution générale

$2xy + \frac{2y}{x} + \frac{-5}{x} = C$

$y(1) = 0$

$0 + 0 - 5 = C$

$C = -5$

Solution unique

$$\boxed{2xy + \frac{2y}{x} - \frac{5}{x} = -5}$$

**Qu. 2.** *Résoudre le problème à valeur initiale.* / Solve the initial value problem.

$$x^2y'' + 5xy' + 4y = 0, \qquad y(1) = 2,\ y'(1) = 0.$$

On pose $y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

$m(m-1) + 5m + 4 = 0$

$m^2 - m + 5m + 4 = 0$

$m^2 + 4m + 4 = 0$

$(m+2)^2 = 0$

$m_1 = m_2 = -2$

Solution générale

$y(x) = C_1 x^{-2} + C_2 x^{-2} \ln x$

$y'(x) = -2C_1 x^{-3} - 2C_2 x^{-3} \ln x + C_2 x^{-3}$

$y(1) = 2 \qquad y'(1) = 0$

① $2 = C_1 \cdot (1)^{-2} + C_2 (1)^{-2} \cancel{\ln 1}^{0}$

$2 = C_1$

$C_1 = 2$

② $0 = -2C_1(1)^{-3} - 2C_2(1)^{-3} \cancel{\ln 1}^{0} + C_2(1)^{-3}$

$0 = -2(2) + C_2$

$C_2 = 4$

$$\boxed{\begin{aligned} &\text{Solution unique} \\ &y(x) = 2x^{-2} + 4x^{-2}\ln x \\ &\quad = \frac{2}{x^2} + \frac{4}{x^2}\ln x \end{aligned}}$$

✓

**Qu. 3.** *Trouver la solution générale.* / Find the general solution.

$$y'' - 2y' - 8y = x - e^{4x}.$$

$\lambda^2 - 2\lambda - 8 = 0$ $\quad P = -8 \quad -4+2$

$(\lambda - 4)(\lambda + 2) = 0$ $\quad S = -2$

$\lambda_1 = 4$

$\lambda_2 = -2$

$y_h = C_1 e^{+4x} + C_2 e^{-2x}$

$y_{p1} = ax + b$

$y'_{p1} = a$

$y''_{p1} = 0$

$0 - 2a \underbrace{-8ax}, -8b = x.$

$-8a = 1$

$\boxed{a = -1/8}$

$-2a - 8b = 0$

$\frac{1}{4} - 8b = 0$

$8b = 1/4$

$\boxed{b = 1/32}$

$$y_{p1} = \frac{-x}{8} + \frac{1}{32}$$

$y_{p2} = -axe^{4x}$

$y'_{p2} = -4axe^{4x} - ae^{4x}$

$y''_{p2} = -16axe^{4x} - 4ae^{4x} - 4ae^{4x}$

$= -16axe^{4x} - 8ae^{4x}$

$-16ax\cancel{e^{4x}} - 8ae^{4x} + 8axe^{4x} + 2ae^{4x} + 8axe^{4x} = -e^{4x}$

$-16\cancel{a}x - 8a + 8\cancel{ax} + 2a + \cancel{8ax} = -1$

$-6a = -1$

$\boxed{a = 1/6}$

$$y_{p2} = \frac{-xe^{4x}}{6}$$

$$y_g = y_h + y_{p1} + y_{p2} = \boxed{C_1 e^{4x} + C_2 e^{-2x} - \frac{x}{8} + \frac{1}{32} - \frac{xe^{4x}}{6}}$$

10

**Qu. 3.** *Trouver la solution générale.* / Find the general solution.

$$y'' - 2y' - 8y = x - e^{4x}.$$

$Ly_h = 0$

$\lambda^2 - 2\lambda - 8 = 0$

$(\lambda - 4)(\lambda + 2) = 0$

$\lambda_1 = 4 \qquad \lambda_2 = -2$

$y_h = C_1 e^{4x} + C_2 e^{-2x}$

$Ly_p = x - e^{4x}$ V.D.P.

$$\begin{bmatrix} e^{4x} & e^{-2x} \\ 4e^{4x} & -2e^{-2x} \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ x - e^{4x} \end{bmatrix}$$

$l_2 = l_2 + 2l_1$

$$\begin{bmatrix} e^{4x} & e^{-2x} \\ 6e^{4x} & 0 \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ x - e^{4x} \end{bmatrix}$$

$C_1' e^{4x} + C_2' e^{-2x} = 0$

$C_2' = -C_1' e^{6x}$

$6e^{4x} C_1' = x - e^{4x}$

$C_1' = \frac{xe^{-4x}}{6} - \frac{1}{6}$

$C_2' = \frac{1}{6}\left[e^{6x} - xe^{2x}\right]$

$u = x \quad du = 1$

$v = \frac{e^{Ax}}{A} \quad dv = e^{Ax}$

$C_2 = \frac{1}{6}\left[\frac{1}{6}e^{6x} - \frac{xe^{2x}}{2} + \frac{e^{2x}}{4}\right]$

$C_2 = \frac{1}{36}e^{6x} - \frac{xe^{2x}}{12} + \frac{e^{2x}}{24}$

$C_1 = \frac{1}{6}\left(\frac{xe^{-4x}}{-4} - \frac{e^{-4x}}{(-4)^2}\right) - \frac{1}{6}x$

$C_1 = \frac{-xe^{-4x}}{24} - \frac{e^{-4x}}{96} - \frac{1}{6}x$

$y_p = \left(\frac{-xe^{-4x}}{24} - \frac{e^{-4x}}{96} - \frac{1}{6}x\right)e^{4x} + \left(\frac{1}{36}e^{6x} - \frac{xe^{2x}}{12} + \frac{e^{2x}}{24}\right)e^{-2x}$

$y_p = -\frac{x}{24} - \frac{1}{96} - \frac{1}{6}xe^{4x} + \frac{1}{36}e^{4x} - \frac{x}{12} + \frac{1}{24}$

$= -\frac{x}{8} + \frac{1}{32} + \frac{1}{36}e^{4x} - \frac{1}{6}xe^{4x}$

déjà inclus dans le 1er terme de $y_h$

$C_1 + \frac{1}{36} = A$

$C_2 = B$

Solution

$$\boxed{y_g = Ae^{4x} + Be^{-2x} - \frac{1}{6}xe^{4x} - \frac{x}{8} + \frac{1}{32}}$$

**Qu. 4.** *Résoudre* / Solve :

$$y' = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix} y, \quad y_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = y(0).$$

$A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$ $\det(A - I\lambda) = 0$ $(A - I\lambda) = \begin{bmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{bmatrix}$ $\det(A - I\lambda) = (-3-\lambda)(-1-\lambda) + 2 = 0$

$= 3 + 3\lambda + \lambda + \lambda^2 + 2 = 0$

$5 + 4\lambda + \lambda^2 = 0$

$\lambda = \dfrac{-4 \pm \sqrt{4^2 - 4(5)}}{2} = \dfrac{-4 \pm 2i}{2} = -2 \pm i$

$\lambda = -2 + i$ $(A - I\lambda)$

$$\begin{bmatrix} -1-i & 2 \\ -1 & 1-i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(-1-i)u_1 + 2u_2 = 0$

$-u_1 + (1-i)u_2 = 0$

$u_1 = (1-i)u_2$

prenons $u_1 = 1-i$, $u_2 = 1$ $u = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

$y = C_1 \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(-2+i)x} = C_1 \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{-2x} e^{ix}$

$= C_1 e^{-2x} \underbrace{\begin{bmatrix} 1-i \\ 1 \end{bmatrix} (\cos(x) + i\sin(x))}$

$\begin{bmatrix} \cos(x) + i\sin(x) - i\cos(x) + \sin(x) \\ \cos(x) + i\sin(x) \end{bmatrix}$

$\begin{bmatrix} \cos(x) + \sin(x) \\ \cos(x) \end{bmatrix} + i \begin{bmatrix} \sin(x) - \cos(x) \\ \sin(x) \end{bmatrix}$

La solution générale est

$y = C_1 e^{-2x} \begin{bmatrix} \cos(x) + \sin(x) \\ \cos(x) \end{bmatrix} + C_2 e^{-2x} \begin{bmatrix} \sin(x) - \cos(x) \\ \sin(x) \end{bmatrix}$

$y_0 \Rightarrow x = 0$

$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$2 = C_1 - C_2$

$-1 = C_1$

$2 = -1 - C_2$

$C_2 = -3$

La solution unique est :

$$\boxed{y = -e^{-2x} \begin{bmatrix} \cos(x) + \sin(x) \\ \cos(x) \end{bmatrix} - 3e^{-2x} \begin{bmatrix} \sin(x) - \cos(x) \\ \sin(x) \end{bmatrix}}$$

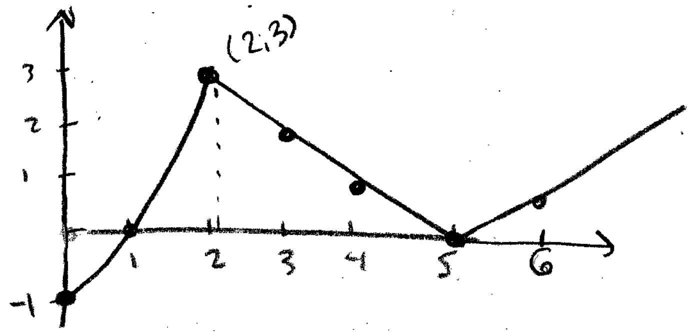
10

**Qu. 5.** *Tracer la fonctions et trouver sa transformée de Laplace.*
Sketch the graph and find the Laplace transform of the function.

$$h(t) = \begin{cases} -1 + t^2, & 0 \le t < 2, \\ 5 - t, & 2 \le t < 5, \\ t - 5, & 5 \le t. \end{cases}$$

$t(0) = -1,\ t(1) = 0,\ t(2) =$
$t(2) = 3,\ t(3) = 2,\ t(4) = 1$
$t(6) = 1$

n.b. $\mathcal{L}\{u(t-a)f(t-a)\}(s) = e^{-as}F(s)$.




$h(t) = (-1 + t^2) - U(t-2)(-1+t^2)$
$+ U(t-2)(5-t) - U(t-5)(5-t)$
$+ U(t-5)(t-5)$

$U(t-2)(-1+t^2) = \boxed{-U(t-2)} + \underline{U(t-2)t^2}$
$U(t-2)t^2 = U(t-2)(t^2-4+4) = U(t-2)(t-2)(t+2) + \boxed{4U(t-2)}$
$= U(t-2)(t-2)(t-2+4) = \boxed{U(t-2)(t-2)(t-2) + 4U(t-2)(t-2)}$
$= 3U(t-2) + U(t-2)(t-2)^2 + 4U(t-2)(t-2)$

$U(t-2)(5-t) = -U(t-2)(t-5) = -U(t-2)(t-5+3-3) = -U(t-2)(t-2) + 3U(t-2)$
$U(t-5)(5-t) = \boxed{-U(t-5)(t-5)}$
$\boxed{U(t-5)(t-5)}$

$h(t) = (t^2-1) - \cancel{3U(t-2)} - U(t-2)(t-2)^2 - 4U(t-2)(t-2) - U(t-2)(t-2) + \cancel{3U(t-2)}$
$+ U(t-5)(t-5) + U(t-5)(t-5)$

$= t^2 - 1 - U(t-2)(t-2)^2 - 5U(t-2)(t-2) + 2U(t-5)(t-5)$

$$\boxed{Y(s) = \frac{2!}{s^3} - \frac{1}{s} - \frac{2!e^{-2s}}{s^3} - \frac{5e^{-2s}}{s^2} + \frac{2e^{-5s}}{s^2}}$$

**Qu. 6.** *Résoudre par transformation de Laplace.* / Solve by Laplace transforms.

$$y'' + 2y' + 5y = \delta(t-3), \qquad y(0) = 0,\ y'(0) = 1.$$

n.b.: $\mathcal{L}\{\delta(t-a)\}(s) = e^{-as}$.

$$\mathcal{L}(y'' + 2y' + 5y) = \mathcal{L}(\delta(t-3))$$

$$Y(s)s^2 - 1 + 2(Y(s)s) + 5Y(s) = e^{-3s}$$

$$Y(s)(s^2 + 2s + 5) = e^{-3s} + 1$$

$$Y(s) = \frac{e^{-3s}}{s^2+2s+5} + \frac{1}{s^2+2s+5}$$

$$= \frac{e^{-3s}}{s^2+2s+1+4} + \frac{1}{s^2+2s+1-1+5}$$

$$Y(s) = \frac{e^{-3s}}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}$$

$a = -1$
$\omega = 2$

$a = 3$

$$\boxed{y(t) = u(t-3)\tfrac{1}{2}\sin 2(t-3)\, e^{-(t-3)} + \tfrac{1}{2}\sin 2t \cdot e^{-t}}$$

✓

**Qu. 7.** *Soit* / Let $f(x) = x^2\, e^x$.
*Par Richardson, extrapoler* $f'(2)$ *obtenue par différence centrée avec* $h = 0.2, 0.1, 0.05$.
By Richardson, extrapolate $f'(2)$ obtained by central difference with $h = 0.2, 0.1, 0.05$.

$$N_1(0.2) = N(0.2) = \frac{1}{0.4}\,[f(2.2) - f(1.8)] = \boxed{60,20051888}$$

$$N_1(0.1) = N(0.1) = \frac{1}{0.2}\,[f(2.1) - f(1.9)] = \boxed{59,38365189}$$

$$N_1(0.05) = N(0.05) = \frac{1}{0.1}\,[f(2.05) - f(1.95)] = \boxed{59,18019874}$$

*Ensuite* / Next

$$N_2(0.2) = N_1(0.1) + \frac{N_1(0.1) - N_1(0.2)}{3} = \boxed{59,11136289}$$

$$N_2(0.1) = N_1(0.05) + \frac{N_1(0.05) - N_1(0.1)}{3} = \boxed{59,11238102}$$

*Enfin* / Finally

$$N_3(0.2) = N_2(0.1) + \frac{N_2(0.1) - N_2(0.2)}{15} = \boxed{59,1124489}$$

✓

*Calculer* $f'(2)$ *exactement et vérifier que* $N_3(0.2)$ *est exacte à 6 décimales près :*
Compute $f'(2)$ exactly and verify that $N_3(0.2)$ is exact to 6 decimals:

$$A := \frac{d}{dx} x^2\, e^x \Big|_{x=2} = \boxed{59,11244879}$$

$$\left|A - N_3(0.2)\right| = \boxed{1,1 \times 10^{-7}} \le \cancel{10^{-6}}\ 10^{-6}.$$

**Qu. 8.** MATLAB ode23 *pour* / for $y' = f(x, y)$, $y(x_0) = y_0$:

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + (1/2)h, y_n + (1/2)k_1), \\ k_3 &= hf(x_n + (3/4)h, y_n + (3/4)k_2), \\ k_4 &= hf(x_n + h, y_n + (2/9)k_1 + (1/3)k_2 + (4/9)k_3), \end{aligned}$$

$$y_{n+1} = y_n + (2/9)k_1 + (1/3)k_2 + (4/9)k_3,$$

*avec estimation de l'erreur locale* / with local error estimate:

$$E_{n+1} = -\frac{5}{72}k_1 + \frac{1}{12}k_2 + \frac{1}{9}k_3 - \frac{1}{8}k_4.$$

*Compléter les 5 petites cases pour l'équadif et les 8* ***grandes cases à 6 décimales***:
Fill-in the 5 the small boxes for the ode and the 7 **long boxes to six decimals**:

$$y' = y^2 + 2y - x, \qquad y(1) = 1, \qquad h = 0.1.$$

*Pour* / For $n = 0$:

$f(x, y) =$ $\boxed{y^2+2y-x}$ $x_0 =$ $\boxed{1}$ $y_0 =$ $\boxed{1}$ $h =$ $\boxed{0,1}$

$k_1 =$ $\boxed{0,2000000}$

$k_2 =$ $\boxed{0,236000}$

$k_3 =$ $\boxed{0,2664329}$

$k_4 =$ $\boxed{0,29244376}$

$y_1 =$ $\boxed{1,2415257}$

$E_1 =$ $\boxed{-0,001174}$

*Pour* / For $n = 1$:

$x_1 =$ $\boxed{1,1}$ $y_1 =$ $\boxed{1,2415257}$ ✓

$k_1 =$ $\boxed{0,29244376}$