D 8.1

mat 2784
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devoir 8

6.48 Résoudre par transformation de Laplace et tracer la solution

$$y'' + 4y = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \geq 1 \end{cases} \qquad \begin{array}{l} y(0)=0 \\ y'(0)=-1 \end{array}$$

$$\mathcal{L}(y) = Y(s)$$
$$\mathcal{L}(y'') = s^2 Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}(y'') + 4\mathcal{L}(y) = s^2 Y(s) - \cancel{sy(0)}^{0} - y'(0) + 4Y(s)$$

$$g(t) = 1 - u \cdot 1(t-1) + u\cancel{(t-1) \cdot 0}^{0}$$

$$s^2 Y(s) + 1 + 4Y(s) = \mathcal{L}(1) - \mathcal{L}(u(t-1))$$
$$Y(s)(s^2+4) + 1 = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$Y(s) = \frac{\frac{1}{s} - \frac{e^{-s}}{s} - 1}{s^2+4}$$

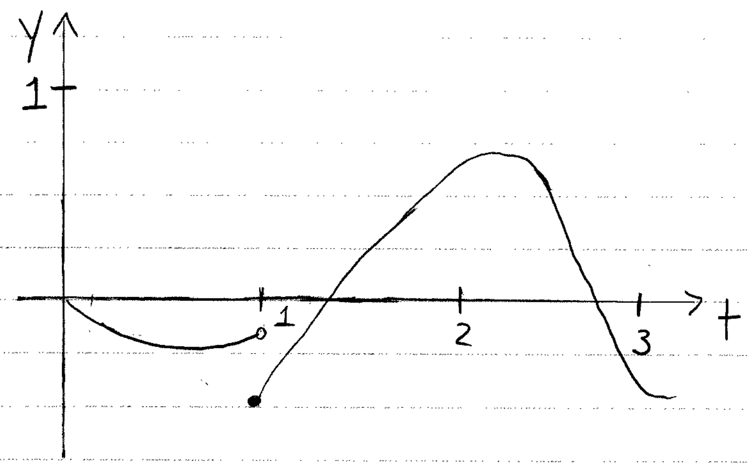
$$Y(s) = \frac{1}{s(s^2+4)} - \frac{e^{-s}}{s(s^2+4)} - \frac{1}{s^2+4}$$

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$
$$1 = A(s^2+4) + Bs^2 + Cs$$
$$1 = As^2 + 4A + Bs^2 + Cs \qquad A+B=0 \qquad C=0 \quad 4A=1$$
$$A = 1/4 \quad B = -1/4 \quad C=0 \qquad 1/4 - 1/4 = 0 \qquad A = 1/4$$
$$B = -1/4$$

$$Y(s) = \frac{1}{4s} - \frac{s}{4(s^2+4)} - e^{-s}\left(\frac{1}{4s} - \frac{s}{4(s^2+4)}\right) - \frac{1}{s^2+4}$$

$$y(t) = \frac{1}{4} - \frac{1}{4}\cos 2t - U(t-1)\left(\frac{1}{4} - \frac{1}{4}\cos 2(t-1)\right) - \frac{1}{2}\sin 2t$$

$$y(t) = \begin{cases} \frac{1}{4} - \frac{1}{4}\cos 2t - \frac{1}{2}\sin 2t & \text{si } 0 \leq t < 1 \\ \frac{-1}{4}\cos 2t + \frac{1}{4}\cos(2t-2) - \frac{1}{2}\sin 2t & \text{si } t \geq 1 \end{cases}$$



6.51) Résoudre le problème par transformation de Laplace et tracer la solution

$$y'' + 4y' = U(t-1) \qquad y(0) = 0 \quad y'(0) = 0$$

on pose $\mathcal{L}(y) = Y(s)$

$\mathcal{L}(y'') = s^2\mathcal{L}(y) - s\,y(0) - y'(0)$

$\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$

$$\mathcal{L}(y'') + 4\mathcal{L}(y') = s^2 Y(s) - \cancel{s\,y(0)}^{0} - \cancel{y'(0)}^{0} + 4s\,Y(s) - \cancel{4y(0)}^{0}$$
$$= s^2 Y(s) + 4s\,Y(s)$$

$$(s^2 + 4s)\,Y(s) = \mathcal{L}(U(t-1))$$
$$Y(s) = \frac{e^{-s}}{s(s^2+4s)} = \frac{e^{-s}}{s^2(s+4)}$$

$$\frac{1}{s^2(s+4)} = \frac{As+B}{s^2} + \frac{C}{s+4}$$

$$1 = As(s+4) + B(s+4) + Cs^2 = As^2 + 4As + Bs + 4B + Cs^2$$

$A+C=0 \qquad -1/16+1/16=0 \qquad C=1/16$

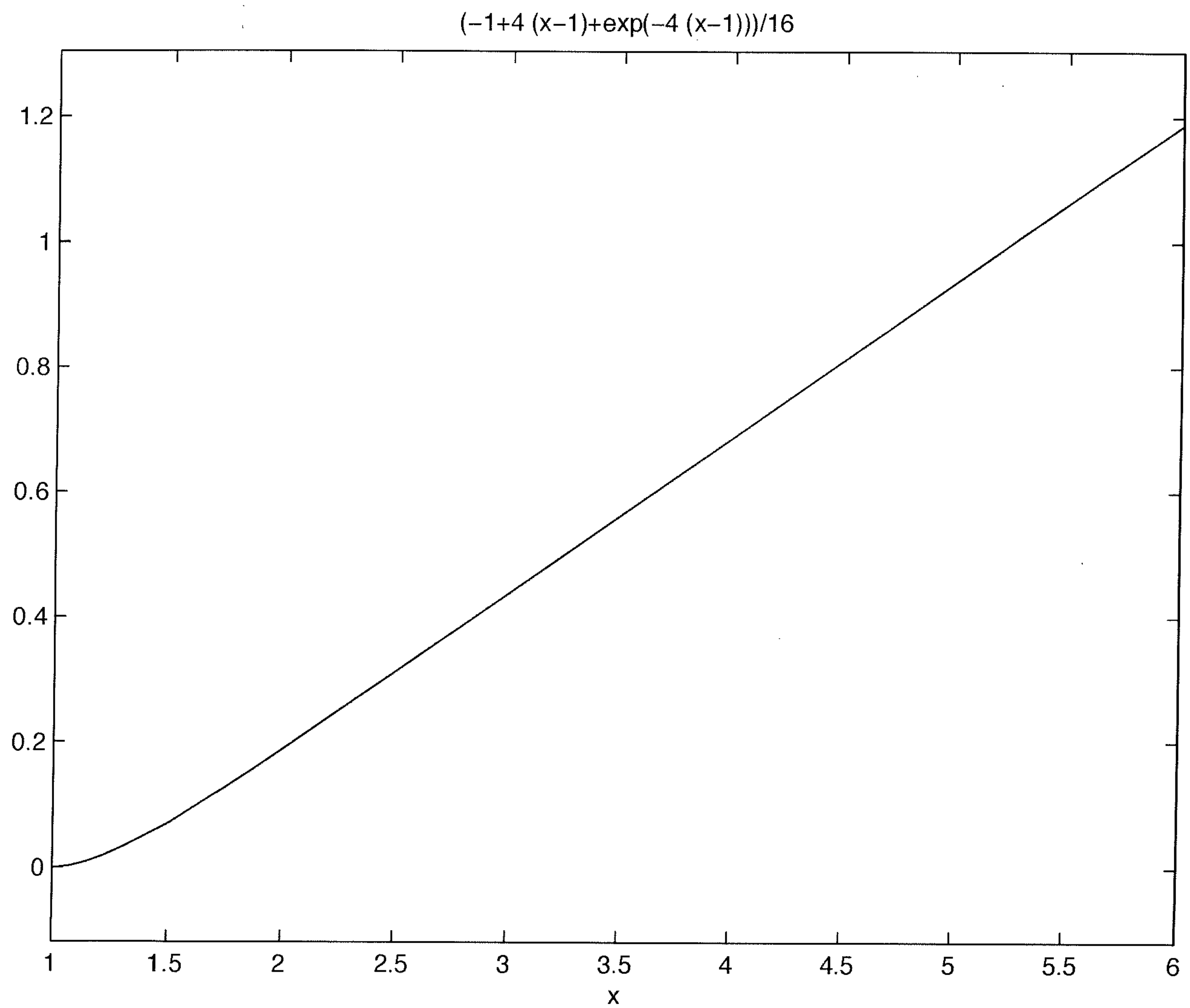
$4A+B=0 \qquad 4\cdot -1/16+1/4=0 \qquad A=-1/16$

$4B=1 \qquad B=1/4$

$$Y(s)=e^{-s}\left(\frac{-1/16\,s+1/4}{s^2}+\frac{1/16}{s+4}\right)$$

$$Y(s)=e^{-s}\left(\frac{-1}{16s}+\frac{1}{4s^2}+\frac{1}{16(s+4)}\right)$$

$$y(t)=u(t-1)\left(\frac{-1}{16}+\frac{1}{4}(t-1)+\frac{1}{16}e^{-4(t-1)}\right)$$



6.53 Résoudre le problème à valeur initiale par transformation de Laplace et tracer la solution

$y'' - y = \sin t + \delta(t - \pi/2) \quad y(0) = 3.5 \quad y'(0) = -3.5$

on pose $\mathcal{L}(y) = Y(s)$

$\mathcal{L}(y'') = s^2 Y(s) - sy(0) - y'(0)$

$g(t) = \sin t + \delta(t - \pi/2)$

$G(s) = \dfrac{1}{s^2+1} + e^{-\pi/2 s}$

$$s^2 Y(s) - 3.5s + 3.5 - Y(s) = \frac{1}{s^2+1} + e^{-\pi/2 s}$$

$$Y(s)(s^2-1) = \frac{1}{s^2+1} + e^{-\pi/2 s} + 3.5s - 3.5$$

$$Y(s) = \frac{1}{(s^2+1)(s^2-1)} + \frac{e^{-\pi/2 s}}{s^2-1} + \frac{3.5s}{s^2-1} - \frac{3.5}{s^2-1}$$

$$\frac{1}{(s^2+1)(s^2-1)} = \frac{A}{s^2+1} + \frac{B}{(s^2-1)}$$ PUISQUE LE 1ER MEMBRE EST PAIRE

$1 = A(s^2-1) + B(s^2+1)$

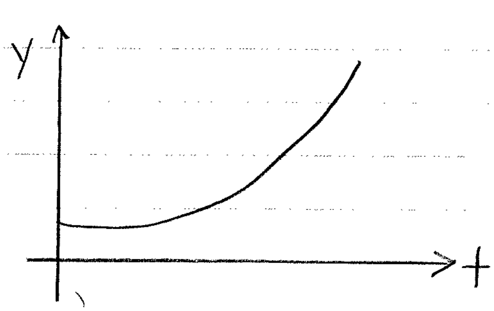
$A + B = 0 \quad B - 1 + B = 0 \quad B = 1/2 \Rightarrow A = -1/2$

$-A + B = 1 \quad A = B - 1$

$$Y(s) = \frac{-1}{2(s^2+1)} + \frac{1}{2(s^2-1)} + \frac{e^{-\pi/2 s}}{s^2-1} + \frac{3.5s}{s^2-1} - \frac{3.5}{s^2-1}$$

$$y(t) = \frac{-1}{2}\sin t + \cancel{\frac{1}{2}\sinh t} + u(t-\pi/2)\sinh(t-\pi/2)$$
$$+ 3.5\cosh t - \cancel{3.5\sinh t}$$

$$y(t) = 3.5\cosh t - 3\sinh t - 1/2\sin t + u(t-\pi/2)\sinh(t-\pi/2)$$



6.57 Résoudre l'équation intégrale par transformée de Laplace et tracer la solution

$$y(t) = \cos 3t + 2\int_0^t y(\tau)\cos 3(t-\tau)\,d\tau$$

$\Rightarrow$ convolution

$$y(t) = \cos 3t + 2\,(y * \cos 3t)$$

$$Y(s) = \frac{s}{s^2+9} + 2\,Y(s)\,\frac{s}{s^2+9}$$

$$Y(s) - 2Y(s)\,\frac{s}{(s^2+9)} = \frac{s}{s^2+9}$$

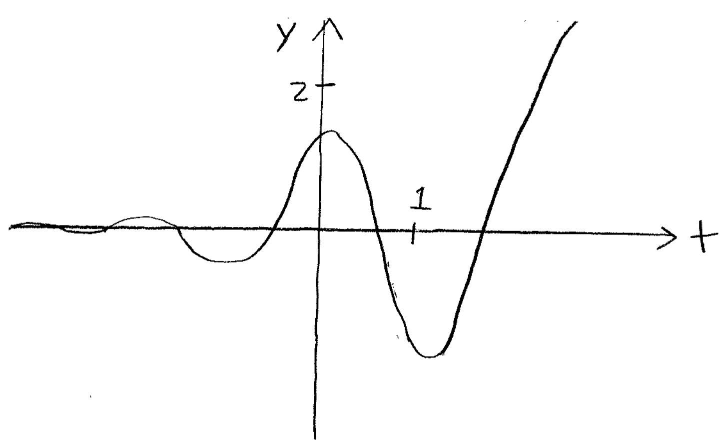
$$Y(s)\left(1 - \frac{2s}{s^2+9}\right) = \frac{s}{s^2+9}$$

$$Y(s)\left(\frac{s^2+9-2s}{s^2+9}\right) = \frac{s}{s^2+9}$$

$$Y(s) = \frac{s}{\cancel{s^2+9}} \cdot \frac{\cancel{s^2+9}}{s^2-2s+9}$$

$$Y(s) = \frac{s-1+1}{(s-1)^2+8} = \frac{s-1}{(s-1)^2+8} + \frac{1}{(s-1)^2+8}$$

$$y(t) = e^{t}\cos(\sqrt{8}t) + \frac{1}{\sqrt{8}}\,e^{t}\sin(\sqrt{8}t)$$



D8.6

6.61 Tracer trois périodes de la fonction 2π-périodique et trouver sa transformée de Laplace

$f(t) = 4\pi^2 - t^2 \quad 0 < t < 2\pi$

$p = 2\pi$

$$\mathcal{L}(f)(s) = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st}(4\pi^2 - t^2)\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left(\int_0^{2\pi} e^{-st}\cdot 4\pi^2 dt - \int_0^{2\pi} e^{-st} t^2 dt \right)$$

$$= \frac{1}{1-e^{-2\pi s}} \left(\frac{4\pi^2}{-s} e^{-st}\Big|_0^{2\pi} - \left(\frac{t^2 e^{-st}}{-s} - \int \frac{2t e^{-st}dt}{-s} \right)\Bigg|_0^{2\pi} \right)$$

$* \; u = t^2 \qquad u' = 2t \qquad\qquad * \; u = t \qquad u' = 1$

$v' = e^{-st}dt \; v = e^{-st}/-s \qquad\qquad v' = e^{-st}dt \; v = e^{-st}/-s$

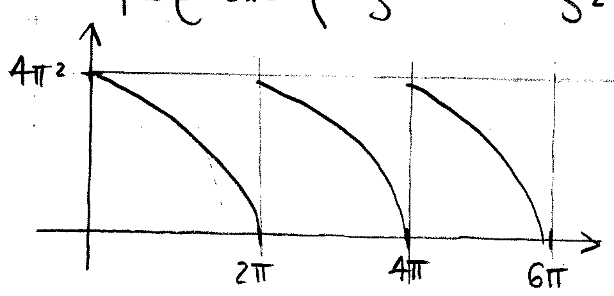
$$= \frac{1}{1-e^{-2\pi s}} \left(\frac{4\pi^2}{-s}\left(e^{-2\pi s} - e^{0}\right) - \left(\frac{t^2 e^{-st}}{-s} + \frac{2}{s}\left(\frac{t e^{-st}}{-s} - \int \frac{e^{-st}}{-s} \right) \right) \right)$$

$$= \frac{1}{1-e^{-2\pi s}} \left(\frac{4\pi^2 e^{-2\pi s}}{-s} - \frac{4\pi^2}{-s} - \left(\frac{t^2 e^{-st}}{-s} + \frac{2}{s}\left(\frac{t e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right) \right)\Bigg|_0^{2\pi} \right)$$

$$= \frac{1}{1-e^{-2\pi s}} \left(\frac{4\pi^2 e^{-2\pi s}}{-s} - \frac{4\pi^2}{-s} - \left(\frac{4\pi^2 e^{-2\pi s}}{-s} + \frac{2}{s}\left(\frac{2\pi e^{-2\pi s}}{-s} - \frac{e^{-2\pi s}}{s^2} \right) + \left(\frac{2}{s}\left(\frac{-e^{+0}}{s^2} \right) \right) \right) \right)$$

$$= \frac{1}{1-e^{-2\pi s}} \left(\cancel{\frac{4\pi^2 e^{-2\pi s}}{-s}} - \frac{4\pi^2}{-s} - \cancel{\frac{4\pi^2 e^{-2\pi s}}{-s}} - \frac{4\pi e^{-2\pi s}}{-s^2} + \frac{2e^{-2\pi s}}{s^3} - \frac{2}{s^3} \right)$$

$$= \frac{1}{1-e^{-2\pi s}} \left(\frac{4\pi^2}{s} + \frac{4\pi e^{-2\pi s}}{s^2} + \frac{2e^{-2\pi s}}{s^3} - \frac{2}{s^3} \right)$$



6.63 tracer 3 périodes de la fonction 2π-périodique et trouver sa transformée de Laplace

$$f(t)=\begin{cases} t & \text{si } 0<t<\pi \\ \pi-t & \text{si } \pi<t<2\pi \end{cases}$$

$$\mathcal{L}(f)(s)=\frac{1}{1-e^{-2\pi s}}\left(\int_0^{\pi} e^{-st}\cdot t\,dt+\int_{\pi}^{2\pi} e^{-st}(\pi-t)\,dt\right)$$

$$=\frac{1}{1-e^{-2\pi s}}\left(\int_0^{\pi} e^{-st}\cdot t\,dt+\int_{\pi}^{2\pi}\pi e^{-st}-\int_{\pi}^{2\pi} e^{-st}\cdot t\,dt\right)$$

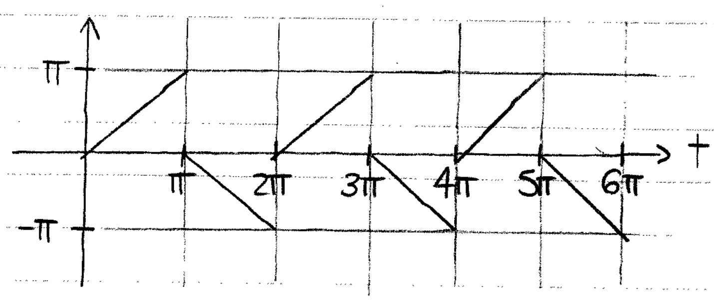
$u=t \qquad u'=1$

$v'=e^{-st}\,dt \qquad v=e^{-st}/-s$

$$=\frac{1}{1-e^{-2\pi s}}\left[\left(\frac{te^{-st}}{-s}-\frac{e^{-st}}{s^2}\right)\Bigg|_0^{\pi}+\frac{\pi e^{-st}}{-s}\Bigg|_{\pi}^{2\pi}-\left(\frac{te^{-st}}{-s}-\frac{e^{-st}}{s^2}\right)\Bigg|_{\pi}^{2\pi}\right]$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\left(\cancel{\frac{\pi e^{-\pi s}}{-s}}-\frac{e^{-\pi s}}{s^2}+\frac{e^{0}}{s^2}\right)+\left(\frac{\pi e^{-2\pi s}}{-s}-\cancel{\frac{\pi e^{-\pi s}}{-s}}\right)-\left(\frac{2\pi e^{-2\pi s}}{-s}-\frac{e^{-2\pi s}}{s^2}-\frac{\pi e^{-\pi s}}{-s}+\frac{e^{-\pi s}}{s^2}\right)\right]$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\frac{1}{s^2}+\frac{\pi e^{-2\pi s}}{s}+\frac{e^{-2\pi s}}{s^2}-\frac{\pi e^{-\pi s}}{s}-\frac{2e^{-\pi s}}{s^2}\right]$$



12.18 $y' = x + \sin y$ $\quad y(0) = 0$ $\quad$ 3 pas avec ODE23 $\quad h = 0{,}1$

$$k_1 = h f(x_n, y_n)$$
$$k_2 = h f(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}k_1)$$
$$k_3 = h f(x_n + \tfrac{3}{4}h,\ y_n + \tfrac{3}{4}k_2)$$
$$k_4 = h f(x_n + h,\ y_n + \tfrac{2}{9}k_1 + \tfrac{1}{3}k_2 + \tfrac{4}{9}k_3)$$

$$y_{n+1} = y_n + \tfrac{2}{9}k_1 + \tfrac{1}{3}k_2 + \tfrac{4}{9}k_3$$

$$E = -\tfrac{5}{72}k_1 + \tfrac{1}{12}k_2 + \tfrac{1}{9}k_3 - \tfrac{1}{8}k_4$$

À $n=0$ $\quad x=0$ $\quad y=0$

$k_1 = 0$
$k_2 = 0{,}005$
$k_3 = 0{,}00787499$
$k_4 = 0{,}010516664$

$y_1 = 0{,}0051667$ ✓
$E = -0{,}0000229$

À $n=1$ $\quad x=0{,}1$ $\quad y=0{,}0051667$

$k_1 = 0{,}010516664$
$k_2 = 0{,}016042481$
$k_3 = 0{,}019219768$
$k_4 = 0{,}022139168$

$y_2 = 0{,}021393316$
$E_2 = -0{,}000025317$

À $n=2$ $\quad x=0{,}2$ $\quad y=0{,}021393316$

$k_1 = 0{,}022139168$
$k_2 = 0{,}02824572$
$k_3 = 0{,}031756474$
$k_4 = 0{,}034982172$

$y_3 = 0{,}049842359$ ✓
$E_3 = -0{,}000027907$

À $n=3$ $\quad x=0{,}3$ $\quad y=0{,}049842359$

$k_1 = 0{,}034982172$
$k_2 = 0{,}041728258$
$k_3 = 0{,}045604955$
$k_4 = 0{,}04916656$

$y_4 = 0{,}091794463$
$E_4 = -0{,}000030566$

12.21 $y' = x + \sin y$ $y(0)=0$ $h=0,1$ ABM3 Erreur en $x=0,5$

$$y_{n+1}^{p} = y_n^c + \frac{h}{12}\left(23 f_n^c - 16 f_{n-1}^c + 5 f_{n-2}^c\right) \qquad f_k^c = f(x_k, y_k^c)$$

$$y_{n+1}^{c} = y_n^c + \frac{h}{12}\left(5 f_{n+1}^p + 8 f_n - f_{n-1}^c\right) \qquad f_k^p = f(x_k, y_k^p)$$

		Départ	Prédite	Corrigée
n	x_n	y_n^c	y_n^p	y_n^c
0	0	0		
1	0,1	0,00516666		
2	0,2	0,021393316		
3	0,3		0,049804504	0,049850703
4	0,4		0,091764516	0,0918126
5	0,5		0,148632615	0,148674842
6	0,6		0,221923701	0,221962429

$$\text{Erreur}(x_n) = \frac{-1}{10}\left[y_{n+1}^c - y_{n+1}^p\right]$$

$$\text{Erreur}(0,5) = 3,87 \times 10^{-6}$$