D 4.1

30/10/2009
(03/11/2009)

MAT2784 - DEVOIR #4

#3.5 Résoudre : $y''' + 12y'' + 36y' = 0$ (EQ HOMOGÈNE)

On pose : $y = e^{\lambda x}$. $D^n = \lambda^n e^{\lambda x}$ $D = \frac{d}{dx}$

$$\lambda^3 e^{\lambda x} + 12\lambda^2 e^{\lambda x} + 36\lambda e^{\lambda x} = 0$$
$$\lambda(\lambda+6)^2 = 0.$$

$\lambda_1 = 0$ $\lambda_{2,3} = -6$

Solution générale : $y = c_1 + c_2 e^{-6x} + c_3 x e^{-6x}$

$y(0) = c_1 + c_2 = 0$ $c_1 = -c_2$

$y'(x) = -6c_2 e^{-6x} + c_3 e^{-6x} - 6c_3 x e^{-6x}$

$y'(0) = -6c_2 + c_3 = 1$ $c_3 = 1 + 6c_2$

$y''(x) = 36c_2 e^{-6x} - 12c_3 e^{-6x} + 36c_3 x e^{-6x}$

$y''(0) = 36c_2 - 12c_3 = -7 \to 36c_2 - 12 - 72c_2 = -7$

$c_2 = -5/36$

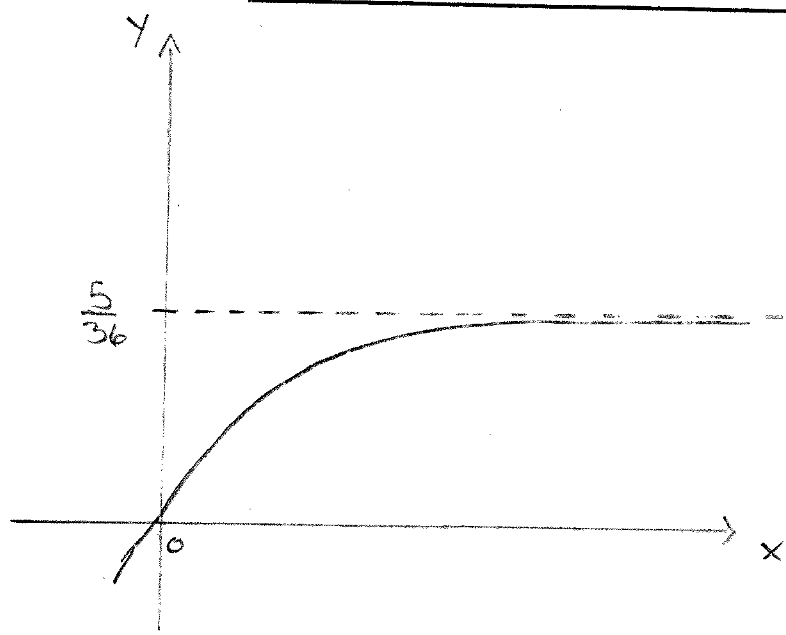
$c_1 = -c_2 = 5/36$

$c_3 = 1 + 6\left(\frac{-5}{36}\right) = 1/6.$

$$y = \frac{5}{36} - \frac{5}{36}e^{-6x} + \frac{1}{6}xe^{-6x}$$

(suite #3.5)

$$\boxed{y = \frac{5}{36} - \frac{5}{36}e^{-6x} + \frac{1}{6}xe^{-6x}}$$



#3.16 Montrer au moyen du WRONSKIEN que :

$$y_1 = x \quad , \quad y_2 = x\ln x \quad , \quad y_3 = x^2 \ln x.$$

sont linéairement indépendants sur $e^{-2} < x < +\infty$

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} x & x\ln x & x^2 \ln x \\ 1 & 1+\ln x & 2x\ln x + x \\ 0 & 1/x & 3 + 2\ln x \end{bmatrix}$$

$$\sim \begin{bmatrix} x & x\ln x & x^2\ln x \\ 0 & 1 & x\ln x + x \\ 0 & 1/x & 3+2\ln x \end{bmatrix} \qquad L_2 = L_2 - \frac{1}{x} L_1$$

$$\begin{bmatrix} x & x\ln x & x^2 \ln x \\ 0 & 1 & x\ln x + x \\ 0 & 0 & \ln x + 2 \end{bmatrix} \qquad L_3 = L_3 - \frac{1}{x} L_2$$

$$\det[\quad] = x(\ln(x) + 2)$$

Racines : $x = 0 \qquad x = e^{-2}.$

Or, les racines sont hors de l'intervalle : $e^{-2} < x < +\infty$

Donc $\omega(x) = x(\ln(x) + 2) \neq 0$ et les fonctions sont linéairement indépendantes

C.Q.F.D

#3.19 Trouver $y_2(x)$ si $y_1(x) = e^x$

$$x(x-2)y'' - (x^2-2)y' + 2(x-1)y = 0$$

$$y_2(x) = u(x)\,y_1(x) = u(x)e^x$$

$$\begin{cases} 2(x-1)\,y_2 = 2(x-1)ue^x \\ (x^2-2)\,y_2' = -(x^2-2)ue^x - (x^2-2)u'e^x \\ x(x-2)\,y_2'' = \underbrace{x(x-2)ue^x}_{uLy_1} + 2x(x-2)u'e^x + x(x-2)u''e^x \end{cases}$$

$$Ly_2 = uLy_1 + \left[-(x^2-2)+2x(x-2)\right]u'e^x + x(x-2)u''e^x$$

$Ly_2 = 0$ et $Ly_1 = 0$

$$0 = (-x^2+2+2x^2-4x)u'e^x + x(x-2)u''e^x$$

$$0 = (x^2-4x+2)u'e^x + x(x-2)u''e^x$$

$$0 = (x^2-4x+2)u' + (x^2-2x)u''$$

$v = u'$ $\quad v' = u''$

$$v(x^2-4x+2) = -(x^2-2)\frac{dv}{dx}$$

$$\int \frac{(x^2-4x+2)}{(2x-x^2)}\,dx = \int \frac{dv}{v}$$

(suite $\rightarrow$)

D 4.5

(#3.19 suite)

$$\ln V = \int \frac{(x^2 - 4x + 2)}{(2x - x^2)}$$

$$\begin{array}{r|l} x^2 - 4x + 2 & -x^2 + 2x \\ \hline -(x^2 - 2x) & -1 \\ -2x + 2 & \end{array}$$

$$\ln V = \int -1 + \frac{2x - 2}{(x^2 - 2x)} \, dx$$

$$\ln V = -x + \ln|x^2 - 2x|$$

$$V = e^{-x} \cdot (x^2 - 2x)$$

$$U' = V = e^{-x}(x^2 - 2x)$$

$$U = \int (x^2 - 2x) e^{-x} dx = \int (x^2 e^{-x} - 2x e^{-x}) dx$$

$$U = -x^2 e^{-x} + \int +2x e^{-x} dx - \int 2x e^{-x} dx$$

$$U = -x^2 e^{-x}$$

$$y_2 = U(x) \, y_1$$

$$y_2 = (-x^2 e^{-x})(e^x)$$

$$\boxed{y_2 = -x^2}$$

#3.22 Résoudre:

$$y'' + 3y' + 2y = 5e^{-2x}$$

$y_g(x) = y_h(x) + y_p(x)$ et $Ly_g = Ly_h + Ly_p = 0 + r(x)$

∘ Solutions homogènes.

Pour $Ly_h = 0 \to \quad y'' + 3y' + 2y = 0$

$$\lambda^2 + 3\lambda + 2 = 0$$
$$(\lambda+2)(\lambda+1) = 0$$

$$\lambda_1 = -2 \qquad \lambda_2 = -1$$

$$y_h = c_1 e^{-2x} + c_2 e^{-x}$$

Solution particulière: $r(x)$ de dimensions finies?

$r(x) = 5e^{-2x} \qquad r'(x) = -10e^{-2x} = -2r(x) \qquad \text{Dim} = 1$

$$y_p(x) = Axe^{-2x}$$
$$y_p'(x) = -2Axe^{-2x} + Ae^{-2x}$$
$$y_p''(x) = 4Axe^{-2x} + -2A\cdot e^{-2x} - 2Ae^{-2x} = 4Axe^{-2x} - 4Ae^{-2x}$$

$$(4Axe^{-2x} - 4Ae^{-2x}) + (-6Axe^{-2x} + 3Ae^{-2x}) + 2Axe^{-2x} = 5e^{-2x}$$
$$\cancel{4Ax} - 4A - \cancel{6Ax} + 3A + \cancel{2Ax} = 5$$
$$A = -5$$

$$y_p = -5xe^{-2x}$$

$$\boxed{y_g = c_1 e^{-2x} + c_2 e^{-x} - 5xe^{-2x}}$$

#3.32 Résoudre

$$y'' + 6y' + 9y = \frac{e^{-3x}}{x^3}$$

non - homogène.
$n = 2$

solution générale $\{ y_g = y_h + y_p$

Sol. Homogène
↳
$$\lambda^2 + 6\lambda + 9 = 0$$
$$(\lambda + 3)^2 = 0.$$
$$\lambda_{1,2} = -3 \quad \rightarrow \quad y_h = c_1 e^{-3x} + c_2 x e^{-3x}$$

Sol Particulière

On cherche $y_p = c_1(x) e^{-3x} + c_2(x) x e^{-3x}$

$$\begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}$$

$$\begin{bmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & e^{-3x} + -3xe^{-3x} \end{bmatrix} \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^{-3x}}{x^3} \end{bmatrix}$$

$$\begin{bmatrix} e^{-3x} & xe^{-3x} \\ 0 & e^{-3x} \end{bmatrix} \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-3x}/x^3 \end{bmatrix}$$

(2) $\quad e^{-3x} c_2'(x) = e^{-3x}/x^3 \qquad c_2'(x) = \frac{1}{x^3}$

(1) $\quad e^{-3x} c_1'(x) + \frac{xe^{-3x}}{x^3} = 0. \qquad c_1'(x) = \frac{-1}{x^2}$

$$c_1(x) = \int -\frac{1}{x^2} dx = 1/x \qquad c_2(x) = \int \frac{1}{x^3} dx = -\frac{1}{2x^2}$$

$$y_p = \frac{1}{x} e^{-3x} + -\frac{1}{2x^2} x e^{-3x} = \frac{1}{2} \frac{e^{-3x}}{x}.$$

$$\boxed{y_g = A e^{-3x} + B x e^{-3x} + \frac{1}{2} \frac{e^{-3x}}{x}}$$

3.37

Résoudre :

$y(1) = e$
$y'(1) = 0$

$$y'' - 2y' + y = \frac{e^x}{x}$$

(Non HOMOGÈNE)

Sol. homogène :

$$\lambda^2 - 2\lambda + 1 = 0$$
$$(\lambda - 1)(\lambda - 1) = 0 \qquad \lambda_{1,2} = 1.$$

$$y_1(x) = e^x \qquad y_2(x) = xe^x$$

Sol particulière. $y_p = c_1(x)\, y_1(x) + c_2(x)\, y_2(x)$.

$$\begin{bmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{bmatrix} \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ e^x/x \end{bmatrix}$$

$$\begin{bmatrix} e^x & xe^x \\ 0 & e^x \end{bmatrix} \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ e^x/x \end{bmatrix}$$

(2) $e^x c_2'(x) = e^x/x \quad \to \quad c_2'(x) = 1/x$.

(1) $e^x c_1'(x) + \frac{\cancel{x}e^x}{\cancel{x}} = 0. \quad \to \quad c_1'(x) = 1$

$c_1(x) = \int -1\, dx = -x \qquad c_2(x) = \int \frac{1}{x}\, dx = \ln x$

$$y_g = Ae^x + Bxe^x - xe^x + xe^x \ln x$$

si $y(1) = e$: $\quad Ae + Be - e + 0 = e.$

$$A + B = 2 \quad \to \quad A = 2 - B$$

$$y'(x) = Ae^x + Be^x + Bxe^x - xe^x - e^x + e^x \ln x + xe^x \ln x + e^x$$

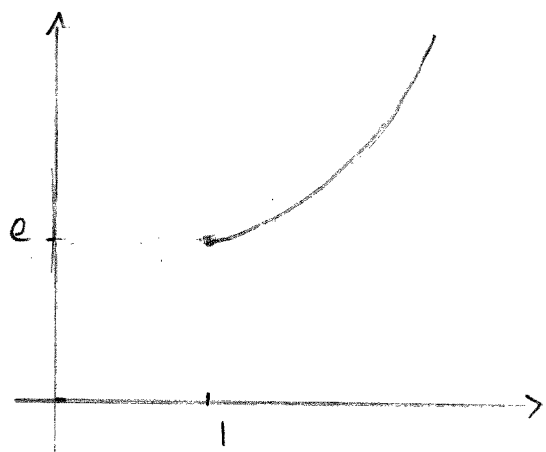
$$y'(1) = Ae + B(2e) - 2e + e = 0$$

(suite) →

(#3:37 suite)

$$Ae + B(2e) = e \qquad \text{or} \qquad A = 2 - B$$
$$2e - Be + B2e = e$$
$$Be = -e$$
$$B = -1 \quad \rightarrow \quad A = 2 + 1 = 3$$

$$y_g = 3e^x - 2xe^x + xe^x \ln x$$



$x \geq 1 \qquad (x_0 \overset{car}{=} 1)$

#9.3 Construire un polynôme de Lagrange qui interpole la fonction $f(x) = e^{2x}\cos 3x$ aux $x_0 = 0$ $x_1 = 0.3$ et $x_2 = 0.6$

$$P_2(x) = f(x_0)\,L_0(x) + f(x_1)\,L_1(x) + f(x_2)\,L_2(x)$$

$$L_0(x) = \frac{(x-0.3)(x-0.6)}{(0-0.3)(0-0.6)} = \frac{(x-0.3)(x-0.6)}{0.18}$$

$$L_1(x) = \frac{(x-0)(x-0.6)}{(0.3-0)(0.3-0.6)} = -\frac{x(x-0.6)}{0.09}$$

$$L_2(x) = \frac{(x-0)(x-0.3)}{(0.6-0)(0.6-0.3)} = \frac{x(x-0.3)}{0.18}$$

$$P_2(x) = \frac{(x-0.3)(x-0.6)}{0.18} - \frac{x(x-0.6)}{0.09}e^{0.6}\cos(0.9) + \frac{x(x-0.3)}{0.18}e^{1.2}\cos(1$$

$$P_2(x) = \frac{x^2 - 0.9x + 0.18}{0.18} - \frac{(x^2 - 0.6x)}{0.09}e^{0.6}\cos(0.9) + \frac{(x^2 - 0.3x)}{0.18}e^{1.2}\cos(1.$$

$$P_2(x) = \left(\frac{1}{0.18} - \frac{1}{0.09}e^{0.6}\cos(0.9) + \frac{1}{0.18}e^{1.2}\cos(1.8)\right)x^2$$

$$+ \left(\frac{-0.9}{0.18} + \frac{0.6}{0.09}e^{0.6}\cos 0.9 - \frac{0.3}{0.18}e^{1.2}\cos(1.8)\right)x$$

$$+ \left(\frac{0.18}{0.18}\right)$$

$$P_2(x) = -11.2202\,x^2 + 3.8082\,x + 1$$

#9.6 Construire le polynôme de Newton aux différences divisées de degré 3 qui interpole les données.

$(-1, 2) \quad (0, 0) \quad (1.5, -1) \quad (2, 4)$

i	x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	-1	$\boxed{2}$			
			$\boxed{-2}$		
1	0	0		$\boxed{0.5\bar{3}}$	
			$-0.\bar{6}$		$\boxed{1.6}$
2	1.5	-1		$5.\bar{3}$	
			10		
3	2	4			

$$P_3(x) = 2 + (x+1)(-2) + x(x+1)(0.5\bar{3}) + x(x+1)(x-1.5)(1.6)$$ ✓

$$P_3(x) = 2 + -2x - 2 + 0.5\bar{3}x^2 + 0.5\bar{3}x + 1.6x^3 + -0.8x^2 - 2.4x$$

$$P_3(x) = 1.6x^3 - 0.2\bar{6}x^2 - 3.8\bar{6}x$$