MAT 2784 A

D1.1

25 sept. 2009

MAT2784A - DEVOIR #1

#1.4. Résoudre :

$$(1+e^x)yy' = e^x$$
$$(1+e^x)y\,dy = e^x dx$$
$$y\,dy = \frac{e^x}{(1+e^x)}dx \qquad u = 1+e^x \quad du = e^x dx$$
$$\int y\,dy = \int \frac{du}{u}$$
$$\frac{1}{2}y^2 = \ln|1+e^x| + C$$

SOL. GÉN. $\quad y^2 = 2[\ln|1+e^x| + C]$ — ceci me suffit

$$y = \pm\sqrt{2[\ln|1+e^x| + C]}$$
$$y = \pm\sqrt{\ln(1+e^x)^2 + K}$$

#1.8. Résoudre :

$$x\sin y\,dx + (x^2+1)\cos y\,dy = 0 \qquad y(1) = \pi/2$$
$$x\sin y\,dx = -(x^2+1)\cos y\,dy$$
$$-\frac{x}{(x^2+1)}dx = \frac{\cos y}{\sin y}dy \qquad u = x^2+1 \quad du = 2x\,dx$$
$$-\frac{1}{2}\int\frac{du}{u} = \int \cot y\,dy$$
$$-\frac{1}{2}\ln|x^2+1| + C_1 = \ln|\sin y|$$
$$\ln(x^2+1)^{-1/2} + C_1 = \ln|\sin y|$$
$$e^{\ln(x^2+1)^{-1/2}+C_1} = e^{\ln|\sin y|}$$
$$C_2(x^2+1)^{-1/2} = \sin y$$

SOL. GÉN. $\quad \sin y = \sqrt{\frac{K}{x^2+1}}$

$$y = \arcsin\sqrt{\frac{K}{x^2+1}}$$

(suite →)

(suite #1.8)

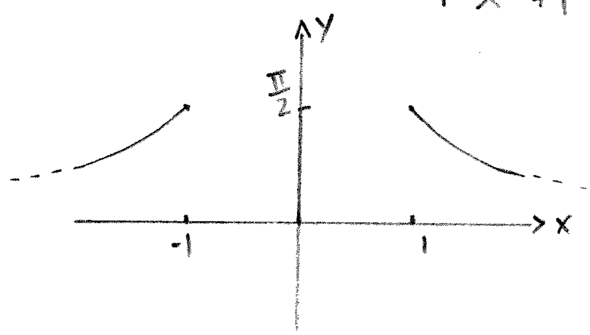
Posons $y = \pi/2$ et $x = 1$

$$\pi/2 = \arcsin\sqrt{\frac{k}{1^2+1}}$$

$$2\sin\pi/2 = k$$

$$k = 2.$$

Donc : $y = \arcsin\sqrt{\frac{2}{x^2+1}}$



1.13 $(2x-5y)dx+(4x-y)dy=0 \quad , y(1)=4$

Démonstration homogène:

$M(\lambda x,\lambda y)=2\lambda x-5\lambda y$
$=\lambda(2x-5)$
deg=1

=

$N(\lambda x,\lambda y)=4\lambda x-\lambda y$
$=\lambda(4x-y)$
deg=1

$y=xu$

$du=xdu+udx$

$(2x-5xy)dx+(4x-xu)(udx+xdu)=0$

$\cancel{x}[(2-5u)dx+(4-u)(udx+xdu)=0$

$(2-5u)dx+(4udx+4xdu-u^2dx-uxdu)=0$

$(2-5u+4u-u^2)dx=(-4x+ux)du$

$(2-5u+4u-u^2)dx=x(-4+u)du$

$\frac{dx}{x}=\frac{-4+u}{-u^2-u+2}$

PLUS SIMPLE D'ÉCRIRE

$\frac{dx}{x}=\frac{4-u}{u^2+u-2}$

Intégration

$-(\ln|x|+c)=\int\frac{u-4}{-u^2-u+2}$

$-(\ln|x|+c)=\int\frac{-1}{u-1}+\frac{2}{u+2}du$

$-(\ln|x|+c)=\ln\left(\frac{(u+2)^2}{u-1}\right)$

$x^{-1}+e^c=\frac{(u+2)^2}{u-1} \quad (k=e^c)$

$\frac{1}{x}=\frac{(u+2)^2}{u-1}$

$x=\frac{u-1}{(u+2)^2}$

$$\boxed{x=\frac{\left(\frac{y}{x}\right)-1}{\left(\frac{y}{x}+2\right)^2}}$$

$\frac{A}{u-1}+\frac{B}{-u-2}=\frac{4-u}{2-u-u^2}$

✓ $4=B-2A \Rightarrow B=-2A-4$

✓ $-1=B-A \Rightarrow B=A-1$

$-4-2A=A-1$

$-3A=3$

✓ $A=-1 \quad B=-2$ ✓

$4-u=(-u-2)A+(u-1)B$

$=(-2A-B)$

$-uA+uB$

$=(B-A)u+(-2A-B)$

(1) $B-A=-1$

(2) $-B-2A=4$

(1)+(2) $-3A=3 \Rightarrow A=-1$

$B=-1+A$

$=-1-1=-2$

#1.19

$$(\sin xy + xy\cos xy)dx + x^2\cos xy\, dy = 0. \quad (*)$$

M_y: $x\cos xy + x\cos xy - x^2y\sin xy$
$= 2x\cos xy - x^2y\sin xy.$

$(M_y = N_x)\Big\{$ $*$ est exacte.

$N_x = 2x\cos xy - x^2y\sin xy$

$u(x,y) = \int N(x,y)dy + T(x)$ DE: $\frac{\partial u}{\partial y} = N$

$= \int x^2\cos xy\, dy + T(x)$

$= x^2\int \cos xy\, dy + T(x)$

$= x\sin xy + T(x)$

u latin
μ grec
ν grec $\neq u$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x\sin xy + T(x))$ DE: $\frac{\partial u}{\partial x} = M.$

$= \sin xy + yx\cos xy + T'(x) = M.$

$\sin xy + yx\cos xy + T'(x) = \sin xy + yx\cos xy$

$T'(x) = 0.$

$T(x) = \int 0\, dx = 0$

$u(x,y) = x\sin xy = K.$

Sol. GÉN.:

$x\sin xy = k.$

C.I. $y(0)=2$

#1.24

$$(2x\cos y+3x^2y)dx+(x^3-x^2\sin y-y)dy=0 \quad (**)$$

$$M_y: -2x\sin y+3x^2$$

$(M_y=N_x)$ $\left\{\begin{array}{l}(**)\text{ est}\\ \text{EXACTE!}\end{array}\right.$

$$N_x: 3x^2-2x\sin y$$

$$u(x,y)=\int M(x,y)\,dx+T(y)$$ DE: $\frac{\partial u}{\partial x}=M.$

$$=\int(2x\cos y+3x^2y)\,dx+T(y)$$

$$=x^2\cos y+x^3y+T(y)$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(x^2\cos y+x^3y+T(y)\right)$$ DE $\frac{\partial u}{\partial y}=N$

$$=-x^2\sin y+x^3+T'(y)=N.$$

$$-x^2\sin y+x^3+T'(y)=x^3-x^2\sin y-y$$

$$T'(y)=\cancel{x^3}\cancel{-x^2\sin y}-y\cancel{-x^3}\cancel{+x^2\sin y}$$

$$=-y.$$

$$T(y)=\int -y\,dy=-y^2/2$$

$$u(x,y):x^2\cos y+x^3y-y^2/2=C.$$

si $y(0)=2 \Rightarrow C=0^2\cos(2)+0^3(2)-(2)^2/2=-2$

$$x^2\cos y+x^3y-y^2/2=-2$$

#8.5. Montrer que la récurrence de point fixe.

$$x_{n+1} = \sqrt{2x_n + 3}$$

pour $f(x) = x^2 - 2x - 3 = 0$ converge sur l'intervalle $[2,4]$

$x_{n+1} = g(x_n) = \sqrt{2x_n+3}$. $a = 2$ et $b = 4$.

- Pour montrer que la récurrence converge sur $[2,4]$, il faut utiliser le théorème du point fixe

1. $g(x) \in [a,b] \ \forall \ x \in [a,b]$

$g(x)$ est croissante sur $[a,b]$ $\Rightarrow 2 \leq g(x) \leq 4$ sur $[a,b]$

deux valeurs ne suffisent pas.

$g(2) = \sqrt{2(2)+3} = \sqrt{7} \approx 2.646$

$g(4) = \sqrt{2(4)+3} = \sqrt{11} \approx 3.317$

Quand $x \in [2,4]$, $g(x) \in [2,4]$

2. $g'(x) \ \exists$ sur $[a,b]$

$$g'(x) = (\sqrt{2x_n+3})' = \frac{1}{\sqrt{2x_n+3}} \rightarrow \text{continue de } [2,4].$$

$g'(x)$ existe sur $[2,4]$

3. $|g'(x)| < 1 \ \forall \ x \in [a,b]$

si $2 < x < 4$??

$$\left.\begin{array}{l} g'(2) = \dfrac{1}{\sqrt{2(2)+3}} = 0.378 < 1 \\ g'(4) = \dfrac{1}{\sqrt{2(4)+3}} = 0.302 < 1 \end{array}\right\} |g'(x)| < 1 \ \forall \ x \in [2,4]$$

$g'(x)$ est décroissante $\Rightarrow 0 < g'(x) \leq 0.3 < 1$

Donc, $x_{n+1} = \sqrt{2x_n+3}$ converge sur $[2,4]$. CQFD sur $[a,b]$

#8.8 $x_{n+1} = g(x) = \cos(x-1)$ $x_0 = 2$ (RADIANS)

n	x_n	
0	2	$x_1 = \cos(x_0 - 1)$
1	0.540302	$x_2 = \cos(x_1 - 1)$
2	0.896187	$x_3 = \cos(x_2 - 1)$
3	0.994616	$x_4 = \cos(x_3 - 1)$
4	0.999985	$x_5 = \cos(x_4 - 1)$
5	0.999999	$x_6 = \cos(x_5 - 1)$
6	1.000000	

ORDRE DE CONVERGENCE:

$$g'(1) = -\sin(1-1) = 0.$$

$$g''(1) = -\cos(1-1) = -1 \neq 0.$$

ORDRE DE CONVERGENCE = 2.

#8.9 $x_{n+1} = g(x) = 1 + \sin^2 x$ $x_0 = 1$ (RADIANS)

n	x_n
0	1,000 000
1	$1 + \sin^2(x_0) =$ 1,708073
2	$1 + \sin^2(x_1) =$ 1,981273
3	$1 + \sin^2(x_2) =$ 1,840761
4	$1 + \sin^2(x_3) =$ 1,928872
5	$1 + \sin^2(x_4) =$ 1,877169 $\approx p$.

ORDRE DE CONVERGENCE?

$$g'(p) = 2\sin(p)\cdot\cos(p) \neq 0$$

∘ Donc ... ORDRE DE CONVERGENCE DE 1.