NOM/ NAME: SOLUTIONS

N° d'Ét/ST#:

Université d'Ottawa · University of Ottawa

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Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 2784 A

Test mi-session 2
Durée: 80 min
Place: LPR 155
16 novembre 2007
10:00–11:20

Midterm 2
Time: 80 min
Place: LPR 155
16th of November 2007
10:00–11:20

Prof.: Rémi Vaillancourt

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
TOTAL	/60

Instructions:

(a) *À livre fermé. Tout type de calculatrices autorisé.*
Closed book. All types of calculators are allowed.

(b) *Répondre sur le questionnaire. Réponses numériques dans les boîtes.*
Answer on the question sheets. Fill-in boxes with numerical answers.

(c) *Les 6 questions sont d'égale valeur.*
All 6 questions have the same value.

(d) *Donner le détail de vos calculs.*
Show all computation.

(e) *Une feuille couleur de tables sera distribuée.*
A one-page table on colored paper will be distributed.

(f) *Tous les angles sont en RADIANS. Tester et ajuster votre calculatrice.*
All angles are in RADIAN measures. Test and adjust your calculators.

$$\sin 1.123456789 = 0.90160112364453$$

Soit les points : / Consider the points:

$$(x_k, f(x_k)), \qquad k = 0, 1, \dots, n.$$

Le polynôme d'interpolation de Lagrange de degré n par ces points :
The nth degree Lagrange interpolation polynomial through these points:

$$p_n(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + \cdots + f(x_n)L_n(x),$$

où / where

$$L_k(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}.$$

Le polynôme d'interpolation newtonien aux différences divisées de degré n par ces points :
The nth degree Newton divided difference interpolation polynomial through these points:

$$p_n(x) = f_0 + (x-x_0)\, f[x_0,x_1] + (x-x_0)(x-x_1)\, f[x_0,x_1,x_2] + \cdots$$
$$+ (x-x_0)(x-x_1)\cdots(x-x_{n-1})\, f[x_0,x_1,\dots,x_n].$$

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Qu. 1. *Trouver la solution générale.*

Find the general solution.

$$y''' + 6y'' = 0.$$

Posons $y = e^{\lambda x}$

$\rightarrow \lambda^3 + 6\lambda^2 = 0$

$\lambda^2(\lambda + 6) = 0$

$\lambda_1 = \lambda_2 = 0 \qquad \lambda_2 = -6$

$\therefore$ La solution générale :

$$\boxed{y(x) = C_1 + C_2 x + C_3 e^{-6x}}$$

Vérification :

$y' = C_2 - 6C_3 e^{-6x}$

$y'' = 36 C_3 e^{-6x}$

$y''' = -216 C_3 e^{-6x}$

$y''' + 6y'' = -216 C_3 e^{-6x} + 6(36 C_3 e^{-6x}) = 0$

COEFFICIENTS INDÉTERMINÉS

✓ **Qu. 2.** *Résoudre le problème à valeur initiale.*
Solve the initial value problem.

$$y''' + y' = x, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0.$$

1) Solution homogène

→ Posons $y = e^{\lambda x}$

$\lambda^3 + \lambda = 0$

$\lambda(\lambda^2 + 1) = 0$

$\lambda_1 = 0 \quad \lambda_1 = i \quad , \lambda_2 = -i$

∴ La solution générale homogène

$y_h = C_1 + C_2 \cos x + C_3 \sin x$

2) Solution particulière

$y_p = ax + b$ ← b fait déjà partie de la solution!

$y_p = ax^2 + bx$

$y_p' = 2ax + b$

$y_p'' = 2a$

$y_p''' = 0$

$y''' + y' = 2ax + b = x$

$\Rightarrow b = 0$

et $a = 1/2$

$y_p = \frac{1}{2}x^2$

p. 3 bis
VARIATION DES PARAMÈTRES

p. 3 tiers
INVERSE DE Y(x)

$y_g = y_h + y_p = y(x)$

$y(x) = C_1 + C_2 \cos x + C_3 \sin x + \frac{1}{2}x^2$

$y'(x) = -C_2 \sin x + C_3 \cos x + x$

$y''(x) = -C_2 \cos x - C_3 \sin x + 1$

$y(0) = 0 = C_1 + C_2 \quad (1)$

$y'(0) = 1 = C_3 \quad (2)$

$y''(0) = 0 = -C_2 + 1 \quad (3)$

de (2) $C_3 = 1$

de (3) $C_2 = 1$

de (1) $C_1 = -1$

∴ La solution unique :

$$\boxed{y(x) = -1 + \cos x + \sin x + \frac{1}{2}x^2}$$

✓

Qu. 1 $\quad y''' + y' = 0 \quad y(0) = 0,\ y'(0) = 1,\ y''(0) = 0$

VAR. DES PAR.

$\lambda^3 + \lambda = \lambda(\lambda^2+1) = 0 \quad \lambda_1 = 0,\ \lambda_2 = i,\ \lambda_3 = -i$

$y_h(x) = A + B\cos x + C\sin x$ $\qquad$ p. 3 h/x

$y_p(x) = c_1(x) + c_2(x)\cos x + c_3(x)\sin x$

$$\begin{bmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{bmatrix}\begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix} \underset{\ell_3+\ell_1}{\approx} \begin{bmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

$\ell_1 \sin x + \ell_2 \cos x \Rightarrow$

$$\begin{bmatrix} \sin x & 0 & 1 \\ 0 & -\sin x & \cos x \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix} \to \boxed{c_1'(x) = x}$$

$\to \sin x\, c_1' + c_3' = 0 \Rightarrow \boxed{c_3' = -x\sin x}$

$-c_2'\sin x + c_3'\cos x = 0 \Rightarrow \boxed{c_2' = -x\cos x}$

$c_1(x) = \dfrac{x^2}{2}$

$c_2(x) = -\int x\cos x\,dx = -x\sin x - \cos x$

$c_3(x) = -\int x\sin x\,dx = x\cos x - \sin x$

$y_p(x) = \dfrac{x^2}{2} - (x\sin x + \cos x)\cos x + (x\cos x - \sin x)\sin x$

$= \dfrac{x^2}{2} - (\cos^2 x + \sin^2 x) = \dfrac{x^2}{2} - 1$ $\quad$ dans $y_h(x)$

Sol. gén.

$y(x) = A + B\cos x + C\sin x + \dfrac{x^2}{2}$

C.I. $\quad$ voir sol. par coeff. indét.

$y(x) = \cos x + \sin x + \dfrac{x^2}{2} - 1.$

$$\boxed{\text{INVERSE DE } Y(x)} \quad \boxed{\text{P. 3 tilde}}$$

Qu. 2

$$A^{-1} = \begin{bmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -\sin x & -\cos x \\ 0 & \cos x & -\sin x \end{bmatrix}$$

$$\begin{bmatrix} c_1' \\ c_2' \\ c_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -\sin x & -\cos x \\ 0 & \cos x & -\sin x \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

$$c_1'(x) = x \quad \Rightarrow c_1(x) = x^2/2$$

$$c_2'(x) = -x\cos x \quad \Rightarrow c_2(x) = -x\sin x - \cos x$$

$$c_3'(x) = -x\sin x \quad \Rightarrow c_3(x) = x\cos x - \sin x$$

$$A^{-1} = \frac{1}{\det A}\, adj'(A) \quad , \quad \det A = 1$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & -\sin x & -\cos x \\ 0 & \cos x & -\sin x \end{bmatrix}$$

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✓ **Qu. 3.** *Trouver la solution générale.*

Find the general solution.

$$y'' - 2y' + y = \frac{e^x}{x}.$$

1) Solution homogène :

Posons $y = e^{\lambda x}$

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

$\therefore y_h = C_1 e^x + C_2 x e^x$

2) Solution particulière

$y_p = C_1(x) e^x + C_2(x) x e^x$

$\Rightarrow \begin{bmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^x/x \end{bmatrix}$

$\sim \begin{bmatrix} e^x & xe^x \\ 0 & e^x \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^x/x \end{bmatrix}$

$\cdot\ e^x C_1' + x e^x C_2' = 0 \quad (1)$

$\cdot\ C_2' \cdot e^x = e^x / x$

$C_2' = 1/x \Rightarrow C_2 = \ln x$ ✓

$(1)\ e^x C_1' + x e^x C_2' = 0$

$C_1' = e^{-x}(-x e^x (1/x))$

$C_1' = -1$

$C_1 = -x$ ✓

$\therefore y_p(x) = -x e^x + \ln x \cdot x e^x$

$y_g = y_h + y_p$

$= C_1 e^x + C_2 x e^x - x e^x + \ln x \cdot x e^x$

$\boxed{y_g(x) = C_1 e^x + C_2 x e^x + \ln x \cdot x e^x}$ ✓

$y' = C_1 e^x + C_2 e^x + C_2 x e^x + e^x + \ln x \cdot e^x + \ln x \cdot x e^x$

$y'' = C_1 e^x + C_2 e^x - C_2 x e^x + e^x + e^x/x + \ln x \cdot e^x + e^x + \ln x \, e^x + \ln x \cdot x e^x$

$= C_1 e^x + 2C_2 e^x + C_2 x e^x + 2e^x + e^x/x + 2\ln x \, e^x + \ln x \cdot x e^x$

Vérification : $y'' - 2y' + y$

$= C_1(\cancel{e^x - 2e^x + e^x}) + C_2(\cancel{2e^x + xe^x - 2e^x - 2xe^x} + xe^x) + 2e^x + e^x/x + 2\ln x \, e^x + \cancel{\ln x \cdot x e^x}$

$- 2(e^x + \ln x \, e^x + \ln x \cdot x e^x) + \cancel{\ln x \cdot x e^x}$

$= e^x/x$ ✓

Qu. 4. *Trouver la solution générale.*
Find the general solution.

$$y' = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix} y.$$

1) $|A - \lambda I| = (-1-\lambda)^2 - 4$
$= \lambda^2 + 2\lambda - 3$
$= (\lambda+3)(\lambda-1)$

$\lambda_1 = -3$

$[A+3I]\vec{u} = \vec{0}$

$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Posons $u_1 = 1$
$\Rightarrow u_2 = -2$
$\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda_2 = 1$

$[A - I]\vec{v} = \vec{0}$

$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Posons $v_1 = 1$
$\Rightarrow v_2 = 2$
$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$y(x) = C_1 e^{-3x}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{x}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\boxed{y(x) = \begin{bmatrix} e^{-3x} & e^{x} \\ -2e^{-3x} & 2e^{x} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}}$$

$y' = -3C_1 e^{-3x}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 e^{x}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad = \begin{bmatrix} -1 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} e^{-3x} & e^{x} \\ -2e^{-3x} & 2e^{x} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$

$= \begin{bmatrix} -3e^{-3x} & e^{x} \\ 6e^{-3x} & 2e^{x} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -3e^{-3x} & e^{x} \\ 6e^{-3x} & 2e^{x} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$

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Qu. 5. *Soit la fonction:* / Consider the function:

$$f(x) = e^{2x}\cos 3x.$$

Construire le polynôme de Lagrange qui interpole $f(x)$ aux noeuds:

Construct the Lagrange polynomial that interpolates $f(x)$ at the nodes:

$$x_0 = 0, \quad x_1 = 0.3, \quad x_2 = 0.6.$$

$f(x_0) = f(0) = 1$

$f(x_1) = f(0{,}3) = 1{,}132647721$

$f(x_2) = f(0{,}6) = -0{,}754337519$

$P_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x)$

$L_0(x) = \dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$; $L_1(x) = \dfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$; $L_2(x) = \dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

$\therefore P_2(x) = \dfrac{(x-0{,}3)(x-0{,}6)}{(-0{,}3)(-0{,}6)} + 1{,}13265\,\dfrac{(x)(x-0{,}6)}{(0{,}3)(0{,}3-0{,}6)} - 0{,}754134\,\dfrac{(x)(x-0{,}3)}{(0{,}6)(0{,}6-0{,}3)}$

$$\boxed{P_2(x) = \frac{50}{9}(x-0{,}3)(x-0{,}6) - 12{,}5850\,x(x-0{,}6) - 4{,}1908\,x(x-0{,}3)}$$

Qu. 6. *Compléter le tableau de différences divisées :*
Complete the divided difference table:

i	x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i-2}, x_{i+3}]$
0	3.2	22.0			
			8.400		
1	2.7	17.8		2.856	
			$\boxed{2,117647}$		−0.528
2	1.0	14.2		$\boxed{7,012123}$	
			$\boxed{6,3421053}$		$\boxed{-4,27364}$
3	4.8	38.3		$\boxed{-10,381436}$	
			$\boxed{-41,4125}$		
4	5.6	5.17			

Construire le polynôme de degré 3 qui interpole les données aux 4 points de $x_0 = 3.2$ à $x_3 = 4.8$.
Construct the cubic interpolating polynomial which interpolates the data at the 4 points $x_0 = 3.2$ to $x_3 = 4.8$.

$$p_3(x) = f_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3]$$

$$p_3(x) = 22,0 + (x - 3,2)(8,4) + (x - 3,2)(x - 2,7)(2,856) + (x - 3,2)(x - 2,7)(x - 1,0)(-0,528)$$

$$\boxed{p_3(x) = 22,0 + 8,4(x - 3,2) + 2,856(x - 3,2)(x - 2,7) - 0,528(x - 3,2)(x - 2,7)(x - 1,0)}$$