MAT 2784A DEV. 9
07.12.01

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Exercice # 6.25 p 296

Trouvons la transformation inverse de Laplace de

$$F(s) = \frac{3s^2+8s+3}{(s^2+1)(s^2+9)} = \frac{3(s^2+1)}{(s^2+1)(s^2+9)} + \frac{8s}{(s^2+1)(s^2+9)}$$

$$= 3 \cdot \frac{1}{s^2+9} + 8 \cdot \frac{s}{(s^2+1)(s^2+9)}.$$

$$\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(3 \times \frac{1}{s^2+9}\right) + \mathcal{L}^{-1}\left(8 \times \frac{s}{(s^2+1)(s^2+9)}\right)$$

$$= 3\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right) + 8\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)(s^2+9)}\right)$$

$$= 3 \cdot \frac{1}{3} \sin 3t + 8 \frac{1}{9-1}(\cos t - \cos 3t)$$

$$\boxed{\mathcal{L}^{-1}(F(s)) = \sin 3t + \cos t - \cos 3t}$$

✓

Exercice #6.50 p297

$$y'' + 4y' + 3y = \begin{cases} 4e^{1-t} & , 0 \le t < 1 \\ 4 & t \geq 1 \end{cases} \qquad y(0) = 0 \, , \, y'(0) = 0$$

on prend $y'' + 4y' + 3y = g(t)$ et posons $y = Y(s)$.

$$s^2 Y(s) - \underbrace{Y(0)}_{=0} - \underbrace{s\,y'(0)}_{=0} + 4s\,Y(s) + 3\,Y(s) = g(s).$$

$$Y(s)\left(s^2 + 4s + 3\right) = g(s)$$

on $g(t) = 4e^{1-t} - 4e^{1-t}\mu(t-1) + 4\mu(t-1)$

$$g(s) = \frac{4e}{s+1} - 4e^{-s}\frac{1}{s+1} + 4\,\frac{e^{-s}}{s}.$$

$$g(s) = \frac{4e}{s+1} - \frac{4e^{-s}}{s+1} + 4\,\frac{e^{-s}}{s}.$$

$$Y(s) = \left(\frac{4e}{s+1} - \frac{4e^{-s}}{s+1} + 4\,\frac{e^{-s}}{s}\right)\frac{1}{s^2+4s+3} \quad \text{avec} \left(\frac{1}{s^2+4s+3} = \frac{1}{(s+1)(s+3)}\right)$$

$$Y(s) = \frac{4e}{(s+1)^2(s+3)} - \frac{4e^{-s}}{(s+1)^2(s+3)} + \frac{4e^{-s}}{s(s+1)(s+3)}$$

on

$$\frac{1}{(s+1)^2(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{s+3}$$

$$\Rightarrow \quad 1 = A(s+1)(s+3) + B(s+3) + C(s+1)^2$$
$$= A(s^2+4s+3) + B(s+3) + C(s^2+2s+1)$$
$$= (A+C)s^2 + (4A+B+2C)s + (3A+3B+C)$$

$\Rightarrow \quad A + C = 0 \quad ; \quad 4A + B + 2C = 0 \quad ; \quad 3A + 3B + C = 1$

$A = -C \qquad\qquad B - 2C = 0 \quad ; \quad -2C + 3B = 1$

$\qquad\qquad\qquad B = 2C \quad ; \quad 4C = 1 \Rightarrow C = \frac{1}{4}$

donc $\begin{cases} A = \frac{-1}{4} \\ B = \frac{1}{2} \\ C = \frac{1}{4} \end{cases}$

D 9.3

alors
$$\frac{1}{(s+1)^2(s+3)} = \frac{-\frac{1}{4}}{(s+1)} + \frac{\frac{1}{2}}{(s+1)^2} + \frac{\frac{1}{4}}{(s+3)}$$
et
$$\frac{1}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$\Rightarrow \quad 1 = A(s+1)(s+3) + B(s+3)s + C(s+1)s$$
$$= A(s^2+4s+3) + B(s^2+3s) + C(s^2+s)$$
$$= (A+B+C)s^2 + (4A+3B+C)s + 3A$$

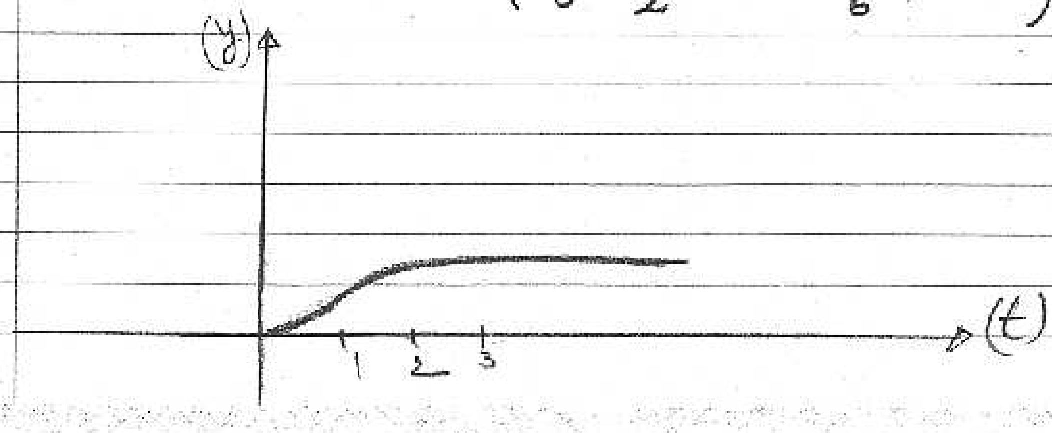
$3A = 1$; $A+B+C=0$; $4A+3B+C=0$.

$A = \frac{1}{3}$ $\quad C = -\frac{1}{3} - B$; $\frac{4}{3} + 3B - \frac{1}{3} - B = 0 \Rightarrow B = -\frac{1}{2}$.

$$\begin{cases} A = \frac{1}{3} \\ B = -\frac{1}{2} \\ C = \frac{1}{6} \end{cases} \Rightarrow \frac{1}{s(s+1)(s+3)} = \frac{\frac{1}{3}}{s} + \frac{-\frac{1}{2}}{s+1} + \frac{\frac{1}{6}}{s+3}$$

donc
$$Y(s) = 4e\left(\frac{-1}{4(s+1)} + \frac{1}{2(s+1)^2} + \frac{1}{4(s+3)}\right) - 4e^{-s}\left(\frac{-1}{4(s+1)} + \frac{1}{2(s+1)^2} + \frac{1}{4(s+3)}\right)$$
$$+ 4e^{-s}\left(\frac{1}{3s} - \frac{1}{2(s+1)} + \frac{1}{6(s+3)}\right).$$

$$\mathcal{L}^{-1}(Y(s)) = y(t) = 4e\left(-\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} + \frac{1}{4}e^{-3t}\right)$$
$$- 4\left(\frac{-1}{4}e^{-(t-1)} + \frac{1}{2}(t-1)e^{-(t-1)} + \frac{1}{4}e^{-3(t-1)}\right)u(t-1)$$
$$+ 4\left(\frac{1}{3} - \frac{1}{2}e^{-(t-1)} + \frac{1}{6}e^{-3(t-1)}\right)u(t-1).$$



Exercice #6.52 p 297

$$y'' + 3y' + 2y = 1 - u(t-1) \quad , \quad y(0) = 0 \; ; \; y'(0) = 1$$

posons $Y(s) = y$. alors

$$\left[s^2 Y(s) + \underbrace{s\,y(0)}_{=0} - \underbrace{y'(0)}_{1}\right] + 3\left[s\,Y(s) - \underbrace{y(0)}_{=0}\right] + 2\,Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$Y(s)\left[s^2 + 3s + 2\right] - 1 = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$Y(s)\left[s^2 + 3s + 2\right] = 1 + \frac{1}{s} - \frac{e^{-s}}{s}$$

$$Y(s) = \left(1 + \frac{1}{s} - \frac{e^{-s}}{s}\right) \times \frac{1}{s^2 + 3s + 2}$$

$$= \left(1 + \frac{1}{s} - \frac{e^{-s}}{s}\right) \times \frac{1}{(s+2)(s+1)}$$

$$Y(s) = \frac{1}{(s+2)(s+1)} + \frac{1}{s(s+2)(s+1)} - \frac{e^{-s}}{s(s+2)(s+1)}$$

Calculons

$$\frac{1}{(s+2)(s+1)} = \frac{A}{(s+2)} + \frac{B}{(s+1)}$$

$$\Rightarrow \quad 1 = A(s+1) + B(s+2)$$

$$= (A+B)s + (A+2B)$$

$$A + B = 0 \; ; \; A + 2B = 1$$

$$A = -B \; ; \; B = 1 \quad et \; A = -1$$

$$\frac{1}{(s+2)(s+1)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+2)}$$

$$\Rightarrow \quad 1 = A(s+1)(s+2) + B\,s(s+2) + C\,s(s+1)$$

$$= A(s^2 + 3s + 2) + B(s^2 + 2s) + C(s^2 + s)$$

$$= (A + B + C)s^2 + (3A + 2B + C)s + 2A$$

$$A + B + C = 0 \; ; \; 3A + 2B + C = 0 \; ; \; 2A = 1 \quad \Rightarrow \quad A = \frac{1}{2}$$

$$C = -\frac{1}{2} - B \; ; \; \frac{3}{2} + 2B - \frac{1}{2} - B = 0 \Rightarrow B = -1$$

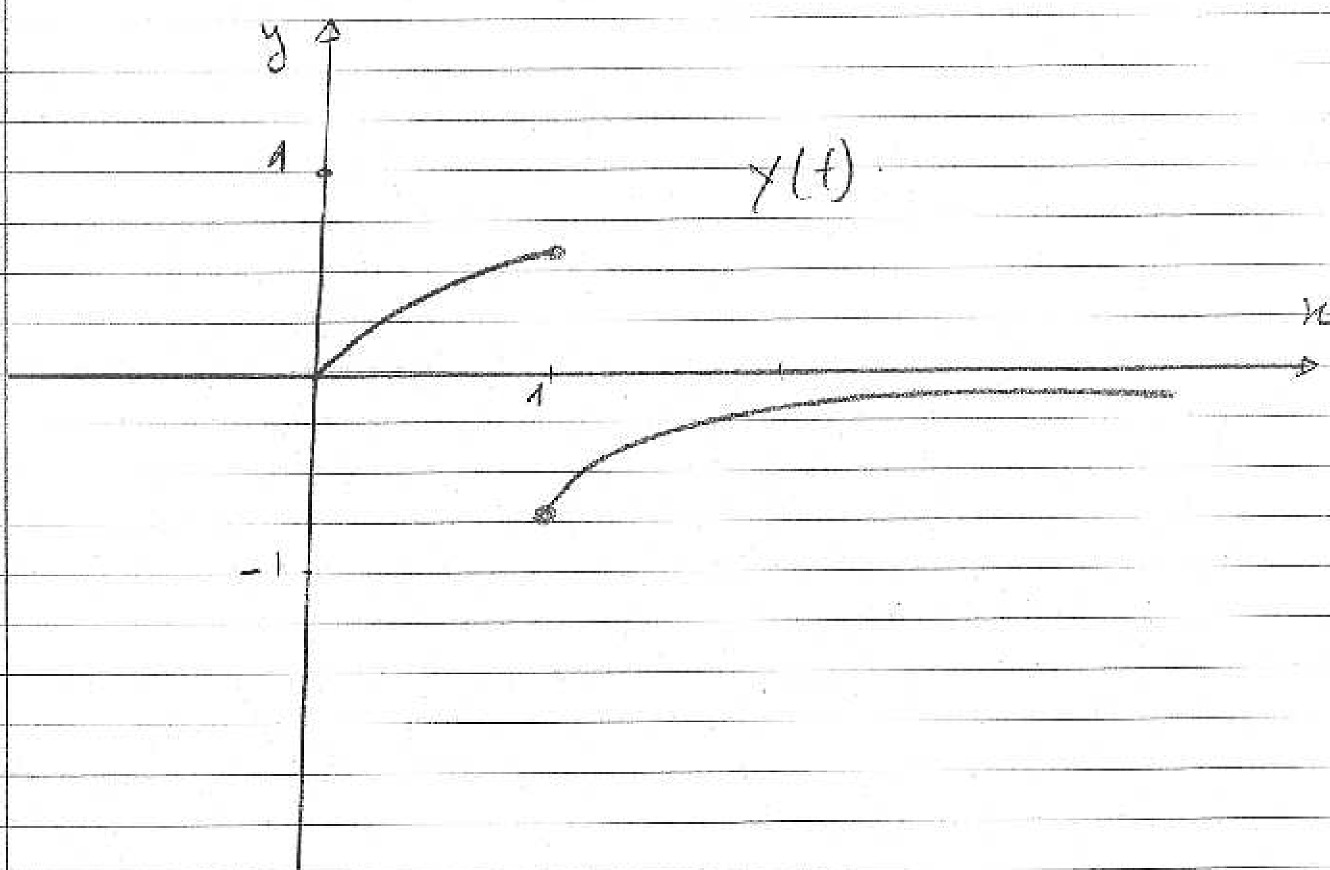
$$\begin{cases} A = \frac{1}{2} \\ B = -1 \\ C = \frac{1}{2} \end{cases} \Rightarrow \frac{1}{s(s+2)(s+1)} = \frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{(s+2)}$$

suite 6.52 D9.5

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)} - e^{-s}\left(\frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}\right)$$

$$\mathcal{L}^{-1}(Y(s)) = y(t) = \cancel{e^{-t}} - e^{-2t} + \frac{1}{2} - \cancel{e^{-t}} + \frac{1}{2}e^{-2t} - u(t-1)\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right)$$

$$y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} - u(t-1)\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right).$$



Exercise # 6.56 p: 297

$$y(t) = \sin(t) + \int_0^t y(\tau)\sin(t-\tau)\,d\tau.$$

$$y(t) = \sin(t) + y \bullet \sin t$$

$$\mathcal{L}(y(t))(s) = \mathcal{L}\left(\sin(t) + y \bullet \sin t\right)$$

$$Y(s) = \frac{1}{s^2+1} + Y(s)\frac{1}{s^2+1}$$

$$Y(s) - Y(s)\frac{1}{s^2+1} = \frac{1}{s^2+1}$$

$$Y(s)\left(\frac{s^2+1-1}{s^2+1}\right) = \frac{1}{s^2+1}$$

$$Y(s)\left(\frac{s^2}{s^2+1}\right) = \frac{1}{s^2+1}$$

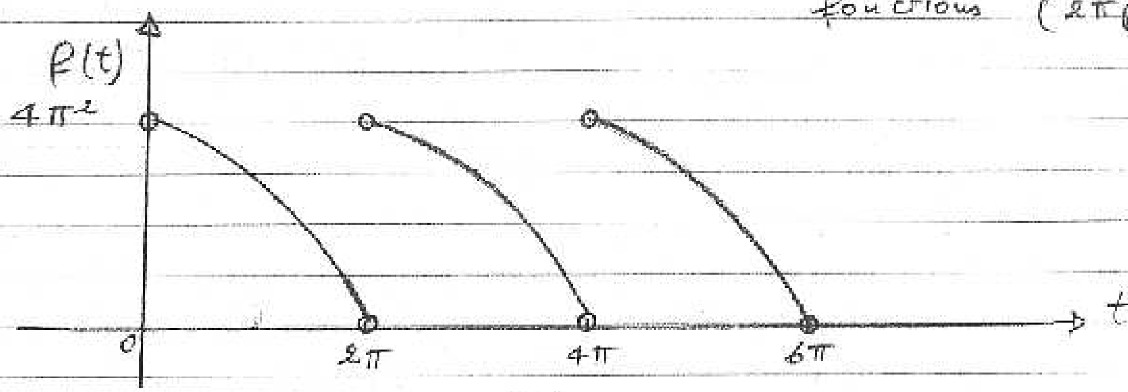
$$Y(s) = \frac{s^2+1}{(s^2+1)s^2} = \frac{1}{s^2}$$

$$\mathcal{L}^{-1}(Y(s))(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$$

$$\boxed{y(t) = t}$$

Exercice # 6.61 p 298

$f(t) = 4\pi^2 - t^2$, $0 < t < 2\pi$ Traçons les 3 périodes de la fonctions (2π période)



$$\mathcal{L}(f(t))(s) = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st}\left(4\pi^2 - t^2\right) dt$$

$$= \frac{1}{1-e^{-2\pi s}} \left[4\pi^2 \int_0^{2\pi} e^{-st}\,dt - \int_0^{2\pi} e^{-st} t^2\,dt \right]$$

Calculons $\int_0^{2\pi} e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_0^{2\pi} = -\frac{1}{s} e^{-2\pi s} + \frac{1}{s} = \frac{1}{s}\left(1 - e^{-2\pi s}\right)$

$U' = e^{-st}$
$V = t^2$

et $\int_0^{2\pi} t^2 e^{-st}\,dt = -\frac{1}{s} e^{-st} t^2 \Big|_0^{2\pi} + \int_0^{2\pi} \frac{1}{s} e^{-st}\, 2t\,dt$ $U' = \frac{1}{s} e^{-st}$
$V = 2t$

$$= -\frac{4\pi^2}{s} e^{-2\pi s} + \left(-\frac{1}{s^2} e^{-st} 2t\right)\Big|_0^{2\pi} + \int_0^{2\pi} \frac{1}{s^2} e^{-st} \times 2\,dt$$

$$= -\frac{4\pi^2}{s} e^{-2\pi s} - \frac{4\pi}{s^2} e^{-2\pi s} + \left(-\frac{2}{s^3} e^{-st}\right)\Big|_0^{2\pi}$$

$$= -\frac{4\pi^2}{s} e^{-2\pi s} - \frac{4\pi}{s^2} e^{-2\pi s} - \frac{2}{s^3} e^{-2\pi s} + \frac{2}{s^3}.$$

donc

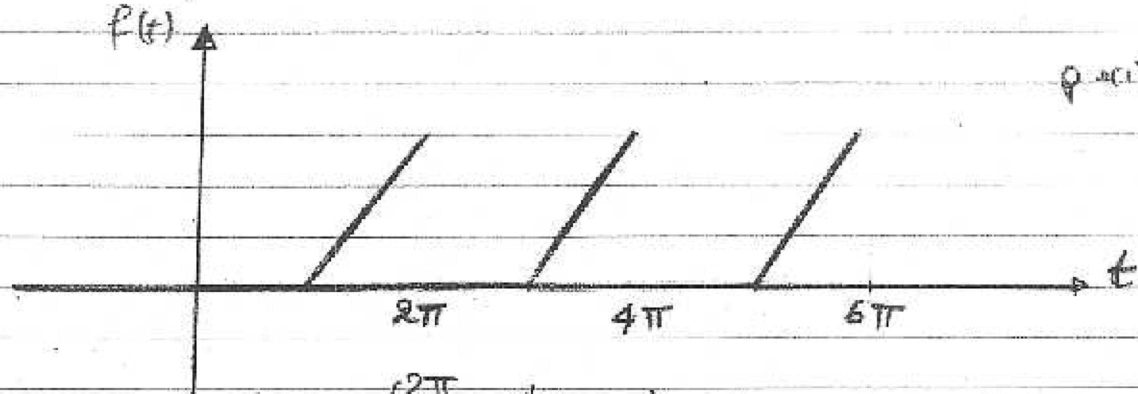
$$\mathcal{L}(f)(s) = \frac{1}{1-e^{-2\pi s}} \left[\frac{4\pi^2}{s}\left(1 - e^{-2\pi s}\right) + \frac{4\pi^2}{s} e^{-2\pi s} + \frac{4\pi}{s^2} e^{-2\pi s} + \frac{2}{s^3} e^{-2\pi s} - \frac{2}{s^3} \right]$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\frac{4\pi^2}{s} + \frac{4\pi}{s^2} e^{-2\pi s} + \frac{2}{s^3} e^{-2\pi s} - \frac{2}{s^3} \right]$$

$$= \frac{\left(\frac{4\pi}{s^2} + \frac{2}{s^3}\right)}{\left(e^{2\pi s} - 1\right)} + \frac{\left(\frac{4\pi^2}{s} - \frac{2}{s^3}\right)}{\left(1 - e^{-2\pi s}\right)}$$

Exercice # 6.64 p: 298 :

$$f(t) = \begin{cases} 0 & si\ 0 < t < \pi \\ t & si\ \pi < T < 2\pi \end{cases}$$



$$F(s) = \frac{1}{1-e^{-2\pi s}} \int_{\pi}^{2\pi} e^{-st}(t-\pi)\, dt.$$

Integration par partie

$$F(s) = \frac{1}{1-e^{-2\pi s}} \left[-\frac{1}{s}(t-\pi) e^{-st} \Big|_{\pi}^{2\pi} + \int \frac{1}{s} e^{-st}\, dt \right]$$

$u = t - \pi \Rightarrow u' = 1$
$v' = e^{-st} \Rightarrow v = -\frac{1}{s} e^{-st}$

$$= \frac{1}{1-e^{-2\pi s}} \left[-\frac{\pi}{s} e^{-2\pi s} + \left(-\frac{1}{s^2} e^{-st} \right) \Big|_{\pi}^{2\pi} \right]$$

$$F(s) = \frac{1}{1-e^{-2\pi s}} \left[-\frac{\pi}{s} e^{-2\pi s} - \frac{1}{s^2} e^{-2\pi s} + \frac{1}{s^2} e^{-\pi s} \right]$$

$$F(s) = \frac{1}{e^{2\pi s} - 1} \left[-\frac{\pi}{s} - \frac{1}{s^2} + \frac{1}{s^2} e^{\pi s} \right]$$

Exercice #12.17 p 307

$y' = x + 2\sin y \qquad y(0) = 0 \qquad$ avec ODE_{23}

$f(x,y) = x + 2\sin y$

$x_0 = 0,\ y_0 = 0$ et $h = 0.1$

Calculons en 6 décimales.

pour $n=0 \quad x_0 = 0$

(1)

$k_1 = 0{,}1\left(0 + 2\sin(0)\right) = 0{,}000000$

$k_2 = 0{,}1\left[0{,}05 + 2\sin(0)\right] = 0{,}005000$

$k_3 = 0{,}1\left[0{,}075 + 2\sin(0 + 0{,}00375)\right] = 0{,}008250$

$k_4 = 0{,}1\left[0{,}1 + 2\sin(0 + 0{,}001667 + 0{,}003667)\right] = 0{,}011067$

$y_1 = 0 + \frac{2}{9}(0) + \frac{1}{3}(0{,}005) + \frac{4}{9}(0{,}008250) = \underline{0{,}005333}$

$EE = -\frac{5}{72}(0) + \frac{1}{12}(0{,}005) + \frac{1}{9}(0{,}008250) - \frac{1}{8}(0{,}011067) = \underline{-0{,}000050}$

pour $n=1 \quad x_1 = 0.1$

(2)

$k_1 = 0{,}1\left(0{,}1 + 2\sin(0{,}005333)\right) = 0{,}011066$

$k_2 = 0.1\left(0{,}15 + 2\sin(0{,}005333 + 0{,}005533)\right) = 0{,}017173.$

$k_3 = 0.1\left(0{,}175 + 2\sin(0{,}005333 + 0{,}012880)\right) = 0{,}021142$

$k_4 = 0.1\left(0{,}2 + 2\sin(0{,}005333 + 0{,}002459 + 0{,}005724 + 0{,}009396)\right) = 0{,}024581$

$y_2 = 0{,}005333 + 0{,}002459 + 0{,}005724 + 0{,}009396 = \underline{0{,}022912}$

$EE = -\frac{5}{72}(0{,}011066) + \frac{1}{12}(0{,}017173) + \frac{1}{9}(0{,}021142) - \frac{1}{8}(0{,}02458) = \underline{-0{,}000061}$

pour $n=2 \quad x_2 = 0{,}2$

(3)

$k_1 = 0{,}1\left(0{,}2 + 2\sin(0{,}022912)\right) = 0{,}024582$

$k_2 = 0{,}1\left(0{,}25 + 2\sin(0{,}022912 + 0{,}012291)\right) = 0{,}032039.$

$k_3 = 0{,}1\left(0{,}275 + 2\sin(0{,}022912 + 0{,}024029)\right) = 0{,}036885.$

$k_4 = 0{,}1\left(0{,}3 + 2\sin(0{,}022912 + 0{,}005463 + 0{,}010680 + 0{,}016393)\right) = 0{,}041084$

$y_3 = 0{,}022912 + 0{,}005463 + 0{,}010680 + 0{,}016393 = \underline{0{,}055448}$

$EE = -\frac{5}{72}(0{,}024582) + \frac{1}{12}(0{,}032039) + \frac{1}{9}(0{,}036885) - \frac{1}{8}(0{,}041084) = \underline{-0{,}000074}$

pour $n=3 \quad x_3 = 0.3$

$k_1 = 0{,}1\left(0{,}3 + 2\sin(0{,}055448)\right) = 0{,}041084.$

$k_2 = 0{,}1\left(0{,}35 + 2\sin(0{,}055448 + 0{,}020542)\right) = 0{,}050183$

$k_3 = 0{,}1\left(0{,}375 + 2\sin(0{,}055448 + 0{,}037638)\right) = 0{,}056090$

$k_4 = 0{,}1\left(0{,}4 + 2\sin(0{,}055448 + 0{,}009130 + 0{,}016728 + 0{,}024929)\right) = 0{,}061207$

$y_4 = 0{,}055448 + 0{,}009130 + 0{,}016728 + 0{,}024929 = \underline{0{,}106235}$

$EE = -\frac{5}{72}(0{,}041084) + \frac{1}{12}(0{,}050183) + \frac{1}{9}(0{,}056090) - \frac{1}{8}(0{,}061207) = \underline{-0{,}000090}$

Exercice # 12.24 p. 307

on a $y' = x + \sin y$ $y(0) = 0$

d'après l'exercice 12.12 on a.

n	x_n	y_n	k_1	k_2	k_3	k_4
0	0,000000	0,000000	0,000000	0,005000	0,005250	0,010525
1	0,100000	0,005171	0,010517	0,016043	0,016319	0,022149
2	0,200000	0,021403	0,022140	0,028247	0,028552	0,034993
3	0,300000	0,049858	0,034984	0,041730	0,042066	0,049179
4	0,400000	0,091817	0,049165	0,056614	0,056984	0,064825
5	0,500000	0,148682				
6	0,600000	0,221963				

Calculons y_5 et y_6 avec ABM4 et estimons l'erreur locale en x_5 et x_6

$$y^p_{n+1} = y^c_n + \frac{h}{24}\left(55 f^c_{n} - 59 f^c_{n-1} + 37 f_{n-2} - 9 f^c_{n-3}\right)$$

$$y^c_{n+1} = y^c_n + \frac{h}{24}\left(9 f^p_{n+1} + 19 f^c_{n} - 5 f^c_{n-1} + f^c_{n-2}\right) \quad \text{avec}$$

$$f^c_n = f(x_n, y^c_n) \quad \text{et} \quad f^p_n = f(x_n, y^p_n)$$

donc

$$y^p_5 = 0,091817 + \frac{0,1}{24}\Big[55(0.4 + \sin(0,091817)) - 59(0.3 + \sin 0,049858) + 37(0.2 + \sin(0,021403)) - 9(0.1 + \sin(0,005171))\Big]$$

$$\underline{y^p_5 = 0,148683}$$

$$y^c_5 = 0,091817 + \frac{0,1}{24}\Big[9(0.5 + \sin(0,148683)) + 19(0.4 + \sin(0,091817)) - 5(0.3 + \sin 0,049858) + (0.2 + \sin(0,021403))\Big]$$

$$\underline{y^c_5 = 0,148682}$$

$$y^p_6 = 0,148682 + \frac{0,1}{24}\Big[55(0,5 + \sin(0,148682)) - 59(0.4 + \sin(0,091817)) + 37(0,3 + \sin(0,049858)) - 9(0,2 + \sin(0,021403))\Big]$$

$$\underline{y^p_6 = 0,221970}$$

$$y^c_6 = 0,148682 + \frac{0,1}{24}\Big[9(0.6 + \sin(0,221970)) + 19(0.5 + \sin(0,148682)) - 5(0.4 + \sin(0,091817)) + (0,3 + \sin(0,049858))\Big]$$

$$\underline{\underline{y^c_6 = 0,221963}}$$