SOLUTIONS 07.1

(1) 13 novembre 2007 | Devoir 7 MAT 2784A | Rémi Vaillancourt

P. 293

5.12

$y'' - xy' - y = 0$

On pose $y = a_0 + a_1 x + a_2 x^2 + \ldots$

$$Ly \begin{cases} y'' & 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3 + \ldots \\ -xy' & -a_1 x - 2a_2 x^2 - 3a_3 x^3 - \ldots \\ -y & -a_0 - a_1 x - a_2 x^2 - a_3 x^3 - \ldots \end{cases}$$

$$0 = (2a_2 - a_0)\cdot 1 + (6a_3 - 2a_1)x + (12a_4 - 3a_2)x^2 + (20a_5 - 4a_3)x^3 + \ldots$$

$2a_2 - a_0 = 0 \Rightarrow a_2 = \frac{a_0}{2}$

$6a_3 - 2a_1 = 0 \Rightarrow a_3 = \frac{a_1}{3}$

$12a_4 - 3a_2 = 0 \Rightarrow a_4 = \frac{1}{4} a_2 = \frac{a_0}{8}$

$20a_5 - 4a_3 = 0 \Rightarrow a_5 = \frac{1}{5} a_3 = \frac{a_1}{15}$

$$y(x) = a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{3} x^3 + \frac{a_0}{8} x^4 + \frac{a_1}{15} x^5 + \ldots$$

$$= \left[a_0 + \frac{a_0}{2} x^2 + \frac{a_0}{8} x^4 + \ldots\right] + \left[a_1 x + \frac{a_1}{3} x^3 + \frac{a_1}{15} x^5 \ldots\right]$$

Solution générale

$$y(x) = a_0 \left[1 + \frac{x^2}{2} + \frac{x^4}{8} + \ldots\right] + a_1 \left[x + \frac{x^3}{3} + \frac{x^5}{15} + \ldots\right]$$

D7.2

2 **Question 5.15**

Solution

$$y''-4xy'+(4x^2-2)y=0$$

$$\begin{aligned}(4x^2-2)\cdot y(x)&=(4x^2-2)(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4)\\&=4x^2a_0-2a_0+4a_1x^3-2a_1x+4a_2x^4-2a_2x^2+4a_3x^5-2a_3x^3+4a_4x^6-2a_4x^4\\&=-2a_0-2a_1x+4x^2a_0-2a_2x^2+4a_1x^3-2a_3x^3+4a_2x^4-2a_4x^4+(\ldots)\end{aligned}$$

$$-4x\cdot y'(x)=-4a_1x-8a_2x^2-12a_3x^3-16a_4x^4-(\ldots)$$

$$1\cdot y''(x)=2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+(\ldots)$$

$$0=Ly=\begin{cases}-2a_0-2a_1x+4x^2a_0-2a_2x^2+4a_1x^3-2a_3x^3+4a_2x^4-2a_4x^4+(\ldots)+\\-4a_1x-8a_2x^2-12a_3x^3-16a_4x^4-(\ldots)\\2a_2+6a_3x+12a_4x^2+20a_5x^3+30a_6x^4+(\ldots)\end{cases}$$

$$0=\begin{cases}(2a_2-2a_0)+(6a_3-6a_1)x+(4a_0-10a_2+12a_4)x^2+\\(4a_1-14a_3+20a_5)x^3+(4a_2-18a_4+30a_6)x^4\end{cases}$$

$$2a_2-2a_0=0\rightarrow a_2=a_0$$

$$6a_3-6a_1=0\rightarrow a_3=a_1$$

$$4a_0-10a_2+12a_4=0\rightarrow 12a_4-6a_0=0\rightarrow a_4=\frac{a_0}{2}$$

$$4a_1-14a_3+20a_5=0\rightarrow 20a_5-10a_1=0\rightarrow a_5=\frac{a_1}{2}$$

$$4a_2-18a_4+30a_6=0\rightarrow 4a_0-18\frac{a_0}{2}+30a_6=0\rightarrow 30a_6-5a_0=0\rightarrow a_6=\frac{a_0}{6}=\frac{a_0}{3!}$$

$$y_1(x)=a_0+a_2x^2+a_4x^4+a_6x^6+a_8x^8,\qquad y_2(x)=a_1x+a_3x^3+a_5x^5+a_7x^7+a_9x^9$$

$$y_1(x)=a_0\left(\frac{1}{0!}+\frac{x^2}{1!}+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}\right),\qquad y_2(x)=a_1\left(\frac{x}{0!}+\frac{x^3}{1!}+\frac{x^5}{2!}+\frac{x^7}{3!}+\frac{x^9}{4!}\right)$$

Solution générale:

$$y(x)=a_0\left(\frac{1}{0!}+\frac{x^2}{1!}+\frac{x^4}{2!}+\frac{x^6}{3!}+\frac{x^8}{4!}\right)+a_1\left(\frac{x}{0!}+\frac{x^3}{1!}+\frac{x^5}{2!}+\frac{x^7}{3!}+\frac{x^9}{4!}\right)$$

(3) #5.27 $f(x) = e^x \quad -1 < x < 1 \qquad P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{1}{2}(3x^2-1)$

On pose $f(x) = \sum_{m=0}^{\infty} a_m P_m(x) \qquad -1<x<1$

donc $a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x) P_m(x)\, dx$

$$a_0 = \frac{1}{2}\int_{-1}^{1} e^x dx = \frac{1}{2}(e - e^{-1}) = 1{,}175$$

$$a_1 = \frac{3}{2}\int x e^x dx = \frac{3}{2}\left(\left[x e^x \right]_{-1}^{1} - \int_{-1}^{1} e^x dx \right) = \frac{3}{2}\left[xe^x - e^x \right]_{-1}^{1}$$

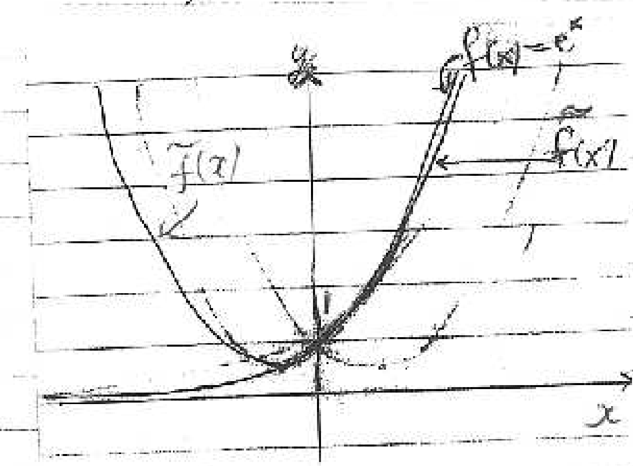
$$= 3e^{-1} = 1{,}104$$

$$a_2 = \frac{5}{4}\int (3x^2-1) e^x dx = \frac{5}{4}\left(3\int x^2 e^x dx - e^x \Big|_{-1}^{1} \right)$$

$$= \frac{5}{4}\left(3\left(\left[x^2 e^x \right]_{-1}^{1} - \int_{-1}^{1} 2x e^x \right) - e^x \Big|_{-1}^{1} \right)$$

$$= \frac{5}{4}\left[3x^2 e^x - 6x e^x + 5e^x \right]_{-1}^{1}$$

$$= \frac{5}{2}(e - 7e^{-1}) = 0{,}3578$$



$$\tilde{f}(x) \approx 1{,}175\, P_0(x) + 1{,}104\, P_1(x) + 0{,}3578\, P_2(x)$$

pas nécessaire
$$= 1{,}175 + 1{,}104\, x + 0{,}5367 x^2 - 0{,}1789$$
$$= 0{,}9961 + 1{,}104 x + 0{,}5367 x^2$$

Hilary

D7.4

(4) <u>Question 5.29</u>

Solution

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x), \qquad -1 < x < 1$$

$$a_m = \frac{2m+1}{2}\int_{-1}^{1} f(x) P_m(x)\,dx$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x) P_0(x)\,dx = \frac{1}{2}\int_{0}^{1} dx = \frac{1}{2}x\Big|_0^1 = \frac{1}{2}$$

$$a_1 = \frac{3}{2}\int_{-1}^{1} f(x) P_1(x)\,dx = \frac{3}{2}\int_{0}^{1} x\,dx = \frac{3}{2}\frac{x^2}{2}\Bigg|_0^1 = \frac{3}{4}$$

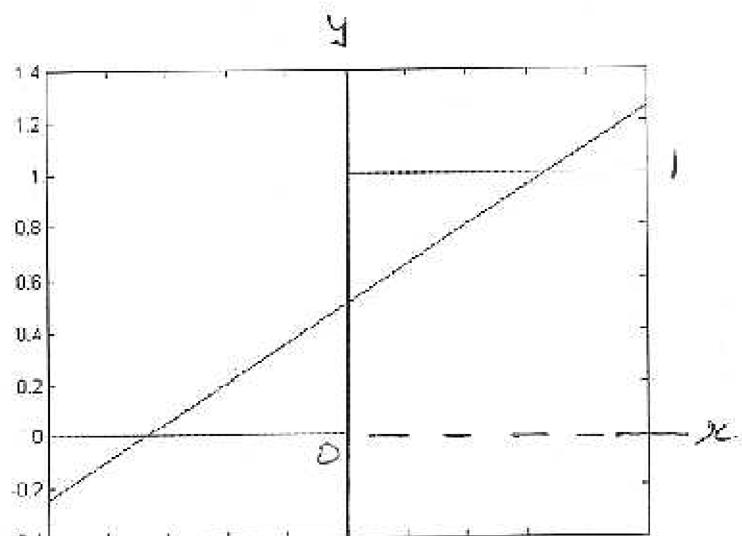
$$a_2 = \frac{5}{2}\int_{-1}^{1} f(x) P_2(x)\,dx = \frac{5}{2}\int_{0}^{1}\frac{1}{2}\left(3x^2-1\right)dx = \frac{5}{4}\left[x^3 - x\right]_0^1 = 0$$

$$a_3 = \frac{7}{2}\int_{-1}^{1} f(x) P_3(x)\,dx = \frac{7}{2}\int_{0}^{1}\frac{1}{2}\left(5x^3-3x\right)dx = \frac{7}{4}\left[\frac{5}{4}x^4 - \frac{3}{2}x^2\right]_0^1 = -\frac{7}{16} \qquad \text{pas demandé}$$

$$f(x) \approx \frac{1}{2}P_0 + \frac{3}{4}P_1 + 0P_2 - \frac{7}{16}P_3$$

$$f_3(x) \approx \frac{1}{2}P_0 + \frac{3}{4}P_1 + 0P_2$$

$$f_3(x) \approx \frac{1}{2} + \frac{3}{4}x$$



D 7.5

(5) <u>Question 5.32</u>

Solution

$$I=\int_{0.3}^{1.7} e^{-x^2}\,dx$$

$$a=0.3$$

$$b=1.7$$

$$f(x)=e^{-x^2}$$

$$\int_a^5 f(x)\,dx=\frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}t+\frac{b+a}{2}\right)dt$$

$$\int_{0.3}^{1.7} e^{-x^2}\,dx=\frac{1.7-0.3}{2}\int_{-1}^{1} e^{-\left(\frac{1.7-0.3}{2}t+\frac{1.7+0.3}{2}\right)^2}dt=0.7\int_{-1}^{1} e^{-(0.7t+1)^2}\,dt$$

$$f(t)=e^{-(0.7t+1)^2}$$

$$\int_{-1}^{1} f(t)\,dx=\frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right)+\frac{8}{9}f(0)+\frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)$$

$$0.7\int_{-1}^{1} e^{-(0.7t-1)^2}\,dt=\frac{5}{9}e^{-\left(-0.7\sqrt{\frac{3}{5}}-1\right)^2}+\frac{8}{9}e^{-(0.7(0)+1)^2}+\frac{5}{9}e^{-\left(0.7\sqrt{\frac{3}{5}}+1\right)^2}$$

$$0.7\int_{-1}^{1} e^{-(0.7t+1)^2}\,dt=\frac{5}{9}e^{-\left(-0.7\sqrt{\frac{3}{5}}+1\right)^2}+\frac{8}{9}e^{-(0.7(0)+1)^2}+\frac{5}{9}e^{-\left(0.7\sqrt{\frac{3}{5}}+1\right)^2}$$

$$\int_{0.3}^{1.7} e^{-x^2}\,dx=0.5803154806$$

D 7.6

6 Question 10.3

Solution

$$\int_0^1 \frac{dx}{1+x}$$

$$n = 10$$

$$h = \frac{1-0}{10} = \frac{1}{10}$$

$$x_0 = 0, \qquad x_1 = \frac{1}{10}, \quad \ldots \quad x_{i-1} = \frac{i-1}{10}, \qquad x_i = \frac{i}{10}, \qquad x_n = 1$$

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\sum_{i=1}^{n}\left[f(x_{i-1}) + f(x_i)\right]$$

$$\int_0^1 \frac{dx}{1+x} \approx \frac{1}{20}\sum_{i=1}^{10}\left[\frac{1}{1+\frac{i-1}{10}} + \frac{1}{1+\frac{i}{10}}\right]$$

$$\int_0^1 \frac{dx}{1+x} \approx 0.6937714032$$

7 Exercice n° 10.4 : Page 230.

$$\int_0^1 \frac{dx}{1+x} \qquad n = 2m = 10$$

$$h = \frac{b-a}{2m} = \frac{1}{10} = 0{,}1$$

$x_0 = 0$; $x_1 = 0{,}1$; $x_2 = 0{,}2$; $x_3 = 0{,}3$; $x_4 = 0{,}4$; $x_5 = 0{,}5$; $x_6 = 0{,}6$; $x_7 = 0{,}7$; $x_8 = 0{,}8$
$x_9 = 0{,}9$; $x_{10} = 1{,}0$.

$$\int_0^1 \frac{dx}{1+x} = \frac{1}{30}\left[1 + \frac{4}{1{,}1} + \frac{2}{1{,}2} + \frac{4}{1{,}3} + \frac{2}{1{,}4} + \frac{4}{1{,}5} + \frac{2}{1{,}6} + \frac{4}{1{,}7} + \frac{2}{1{,}8} + \frac{4}{1{,}9} + 0{,}5\right]$$

$$\int_0^1 \frac{dx}{1+x} = 0{,}693150 .$$

MAT 2784 A D7.7

8

Question 12.2

Solution

$y' = x + \sin y, \qquad y(0) = 0, \qquad h = 0.1, \qquad (x_0 = 0) \le x \le (x_N = 1)$

$f(x, y) = x + \sin y$

$x_n = x_0 + hn = 0.1n$

$y(x_{n+1}) = y_n + h(0.1n + \sin y_n)$

n	xn	yn
0	0,0	0
1	0,1	0,01000
2	0,2	0,03100
3	0,3	0,06410
4	0,4	0,11051
5	0,5	0,17153
6	0,6	0,24860
7	0,7	0,34321
8	0,8	0,45686
9	0,9	0,59097
10	1,0	0,74669