D6.1

SOLUTIONS

6 novembre 2007 — Devoir 6 — MAT 2784A — RÉMI VAILLANCOURT

4.4 $y' = Ay$

$$A = \begin{bmatrix} -1 & 1 & 4 \\ -2 & 2 & 4 \\ -1 & 0 & 4 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 1 & 4 \\ -2 & 2-\lambda & 4 \\ -1 & 0 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \big((-8+2\lambda)-(-4)\big)\cdot 1 - (2-\lambda)\big((4-\lambda)^2-(-4)\big) + 0 = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda + 4 = 0$$

$$(\lambda-1)(\lambda-2)(\lambda-2) = 0 \qquad \lambda_1 = 1 \quad \lambda_2 = \lambda_3 = 2$$

$$(A-I) = \begin{bmatrix} -2 & 1 & 4 \\ -2 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad e_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$(A-2I) = \begin{bmatrix} -3 & 1 & 4 \\ -2 & 0 & 4 \\ -1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad e_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution générale

$$y = c_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} e^{x} + c_2 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{2x} + c_3 \left(x \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) e^{2x}$$

4.8 $y' = Ay + f(x)$

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \qquad f(x) = \begin{bmatrix} 2e^{-x} \\ 3x \end{bmatrix}$$

$$\det(A - \lambda I) = (-2-\lambda)(-2-\lambda) - 1 \cdot 1 = \lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+1)(\lambda+3) = 0 \qquad \lambda_1 = -1 \quad \lambda_2 = -3$$

$$(A+I) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (A+3I) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$y_h(x) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-x} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3x}$$

$$y_p(x) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} xe^{-x} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{-x} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} x + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$y_p'(x) = -\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} xe^{-x} + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{-x} - \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} e^{-x} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$-\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} xe^{-x} + \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \end{bmatrix} e^{-x} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2a_1 + a_2 \\ a_1 - 2a_2 \end{bmatrix} xe^{-x} + \begin{bmatrix} -2b_1 + b_2 \\ b_1 - 2b_2 \end{bmatrix} e^{-x} + \begin{bmatrix} -2d_1 + d_2 \\ d_1 - 2d_2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-x} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} x$$

$-a_1 = -2a_1 + a_2 \Rightarrow a_1 = a_2 = a$ posons $a_1 = 1 \Rightarrow a = 1$

$$\left.\begin{array}{l} a_1 - b_1 = -2b_1 + b_2 + 2 \\ a_2 - b_2 = b_1 - 2b_2 + 0 \end{array}\right\} \left.\begin{array}{l} a_1 + b_1 = b_2 + 2 \\ a_2 + b_2 = b_1 \end{array}\right\} b_1 = b_2 + 1 \qquad \text{posons } b_1 = 2 \Rightarrow b_2 = 1$$

$$\left.\begin{array}{l} 0 - 2c_1 + c_2 = 0 \\ 3 - 2c_2 + c_1 = 0 \end{array}\right\} \left.\begin{array}{l} c_2 = 2c_1 \\ c_1 = 2c_2 - 3 \end{array}\right\} \begin{array}{l} c_1 = 1 \\ c_2 = 2 \end{array}$$

$$\left.\begin{array}{l} c_1 = -2d_1 + d_2 \\ c_2 = -2d_2 + d_1 \end{array}\right\} \left.\begin{array}{l} d_1 = 2d_2 + 2 \\ d_2 = 2d_1 + 1 \end{array}\right\} d_1 = \frac{-4}{3} \quad d_2 = \frac{-5}{3}$$

Solution générale

$$y = c_1\begin{bmatrix}1\\1\end{bmatrix}e^{-x} + c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-3x} + \begin{bmatrix}1\\1\end{bmatrix}xe^{-x} + \begin{bmatrix}2\\1\end{bmatrix}e^{-x} + \begin{bmatrix}1\\2\end{bmatrix}x - \frac{1}{3}\begin{bmatrix}4\\5\end{bmatrix}$$

4.10 $y' = Ay + f(x)$

$$A = \begin{bmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{bmatrix} \qquad f(x) = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(-1-\lambda) - 3 = 0$$

$$\lambda^2 - 4 = 0 \quad (\lambda-2)(\lambda+2) = 0 \quad \lambda_1 = 2 \quad \lambda_2 = -2$$

$$(A - 2I) = \begin{bmatrix}-1 & \sqrt{3}\\ \sqrt{3} & -3\end{bmatrix} \sim \begin{bmatrix}1 & -\sqrt{3}\\ 0 & 0\end{bmatrix} \qquad u = \begin{bmatrix}\sqrt{3}\\1\end{bmatrix}$$

$$(A + 2I) = \begin{bmatrix}3 & \sqrt{3}\\ \sqrt{3} & 1\end{bmatrix} \sim \begin{bmatrix}3 & \sqrt{3}\\ 0 & 0\end{bmatrix} \qquad v = \begin{bmatrix}1\\-\sqrt{3}\end{bmatrix}$$

$$y_h(x) = c_1\begin{bmatrix}\sqrt{3}\\1\end{bmatrix}e^{2x} + c_2\begin{bmatrix}1\\-\sqrt{3}\end{bmatrix}e^{-2x} \qquad y_p(x) = \begin{bmatrix}a\\b\end{bmatrix} \quad y_p'(x) = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}1 & +\sqrt{3}\\ \sqrt{3} & -1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} \qquad \left.\begin{aligned}a + \sqrt{3}b + 1 &= 0\\ \sqrt{3}a - b &= 0\end{aligned}\right\} \begin{aligned}a + \sqrt{3}b &= -1\\ a &= \frac{b}{\sqrt{3}}\end{aligned}$$

$$b = \frac{-\sqrt{3}}{4} \quad a = \frac{-1}{4}$$

Solution générale:

$$y = c_1\begin{bmatrix}\sqrt{3}\\1\end{bmatrix}e^{2x} + c_2\begin{bmatrix}1\\-\sqrt{3}\end{bmatrix}e^{-2x} - \frac{1}{4}\begin{bmatrix}1\\\sqrt{3}\end{bmatrix}$$

4.13 $y' = Ay$ $\quad y(0) = y_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

Par la solution de 4.10 nous savons:

$$y(x) = c_1 \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} e^{2x} + c_2 \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} e^{-2x}$$

$$y(0) \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$$

$$\left.\begin{array}{l} \sqrt{3}c_1 + c_2 = 1 \\ c_1 - \sqrt{3}c_2 = 0 \end{array}\right\} \quad c_1 = \sqrt{3}c_2 \Rightarrow c_2 = \frac{1}{4} \text{ et } c_1 = \frac{\sqrt{3}}{4}$$

<u>Solution unique</u>:

$$y = \frac{\sqrt{3}}{4} \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix} e^{2x} + \frac{1}{4} \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix} e^{-2x}$$

D6.5

5.3 $\sum_{n=2}^{\infty} \frac{\ln(n)}{n} x^n \qquad \frac{1}{R} = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$\left| \frac{\ln(n+1)}{(n+1)} \cdot \frac{n}{\ln(n)} \right| = \left| \frac{n \ln(n+1)}{(n+1)\ln(n)} \right|$$

$$\lim_{n\to\infty} \left| \frac{n \ln(n+1)}{(n+1)\ln(n)} \right| \overset{\boxed{R.H.}}{=} \lim_{n\to\infty} \left| \frac{\frac{n}{n+1}}{\frac{n+1}{n}} \right| = \lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2} + \frac{2n}{n^2} + \frac{1}{n^2}} \right| = 1 \quad \Rightarrow R = 1 \qquad |x| < 1$$

Intervalle de convergence : $-1 < x < 1$

$$\sum_{n=2}^{\infty} \frac{\ln(n)}{x} x^n \qquad \frac{1}{R'} = \lim_{n\to\infty} \left| \frac{\ln(n+1)}{x} \cdot \frac{x}{\ln(n)} \right|$$

$$\lim_{n\to\infty} \left| \frac{\ln(n+1)}{\ln(n)} \right| \overset{\boxed{R.H.}}{=} \lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \left| \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} \right| = 1$$

$\Rightarrow R' = 1 \qquad |x| < 1$

Intervalle de convergence : $-1 < x < 1$

5.4 $\sum_{n=1}^{\infty} \frac{1}{n^2+1} (x+1)^n$

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{n^2+1}{(n+1)^2+1} \right| = \lim_{n\to\infty} \left| \frac{\frac{n^2}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{2}{n^2}} \right| = 1$$

$\Rightarrow R=1 \qquad |x+1|<1$

Intervalle de convergence : $-1<x+1<1 \Rightarrow -2<x<0$

$$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)(x+1)} (x+1)^n$$

$$\frac{1}{R'} = \lim_{n\to\infty} \left| \frac{(n+1)}{((n+1)^2+1)\cancel{(x+1)}} \cdot \frac{(n^2+1)\cancel{(x+1)}}{n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^3+n^2+n+1}{n^3+2n^2+2n} \right| = \lim_{n\to\infty} \left| \frac{1+\frac{1}{n}+1/n^2+1/n^3}{1+2/n+2/n^2} \right| = 1$$

$\Rightarrow R'=1 \qquad |x+1|<1$

Intervalle de convergence : $-1<x+1<1 \Rightarrow -2<x<0$

10.1 pour (10.6)

(10.6)

$$f'(x) = \frac{1}{12h}\left[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)\right] + \frac{h^4}{30} f^{(5)}(\xi)$$

$$f(x) = \cosh(x) - \sinh(x) = \frac{e^x+e^{-x}}{2} - \frac{e^x-e^{-x}}{2} = e^{-x}$$

a) $$f'(1{,}2) = \frac{1}{12h}\left[e^{-1} - 8e^{-1{,}1} + 8e^{-1{,}3} - e^{-1{,}4}\right]$$

$$= -0{,}301193206 = dfn$$

b) $f'(x) = -e^{-x}$

$$f'(1{,}2) = -e^{-1{,}2} = -0{,}30119421 = dfe$$

c) $\varepsilon = dfn - dfe = 0{,}000001005 = 1{,}005 \times 10^{-6}$

d) $|\varepsilon| \leq \frac{h^4}{30}\left|f^{(5)}(\xi)\right|$ $\quad x_0 - 2h \leq \xi \leq x_0 + 2h$; $f^{(5)}(x) = -e^{-x}$; $x_0 = 1{,}2$

$$|\varepsilon| \leq \frac{(0{,}1)^4}{30}\left|-e^{-1{,}0}\right|$$

$$\leq 1{,}2263 \times 10^{-6}$$

$1{,}005 \times 10^{-6}$ est plus petit que $1{,}2263 \times 10^{-6}$ donc l'erreur est vérifié.

MAT 2784 A

D6.8

10.7 $\int_0^1 \frac{1}{1+x^3}dx$ à 10^{-4} près

Methodes des trapèzes

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\dots+f(x_n)\right]-\frac{(b-a)h^2}{12}f''(\xi)$$

$$a<\xi<b$$

$f'(x) = \frac{-3x^2}{(1+x^3)^2}$ $\qquad f''(x) = \frac{18x^4}{(1+x^3)^3} - \frac{6x}{(1+x^3)^2}$

$M = \max_{0\le x\le 1} |f''(x)| \approx 1,75$ en $x \approx 0,4$

v. graphique

$\frac{(1-0)}{12} h^2 (1,75) \le 10^{-4} \Rightarrow h \le 0,0262$

$\frac{1}{h} = 38.1881 \le n$ $\qquad \boxed{n = 39}$

$h = \frac{1}{39} = 0.256$

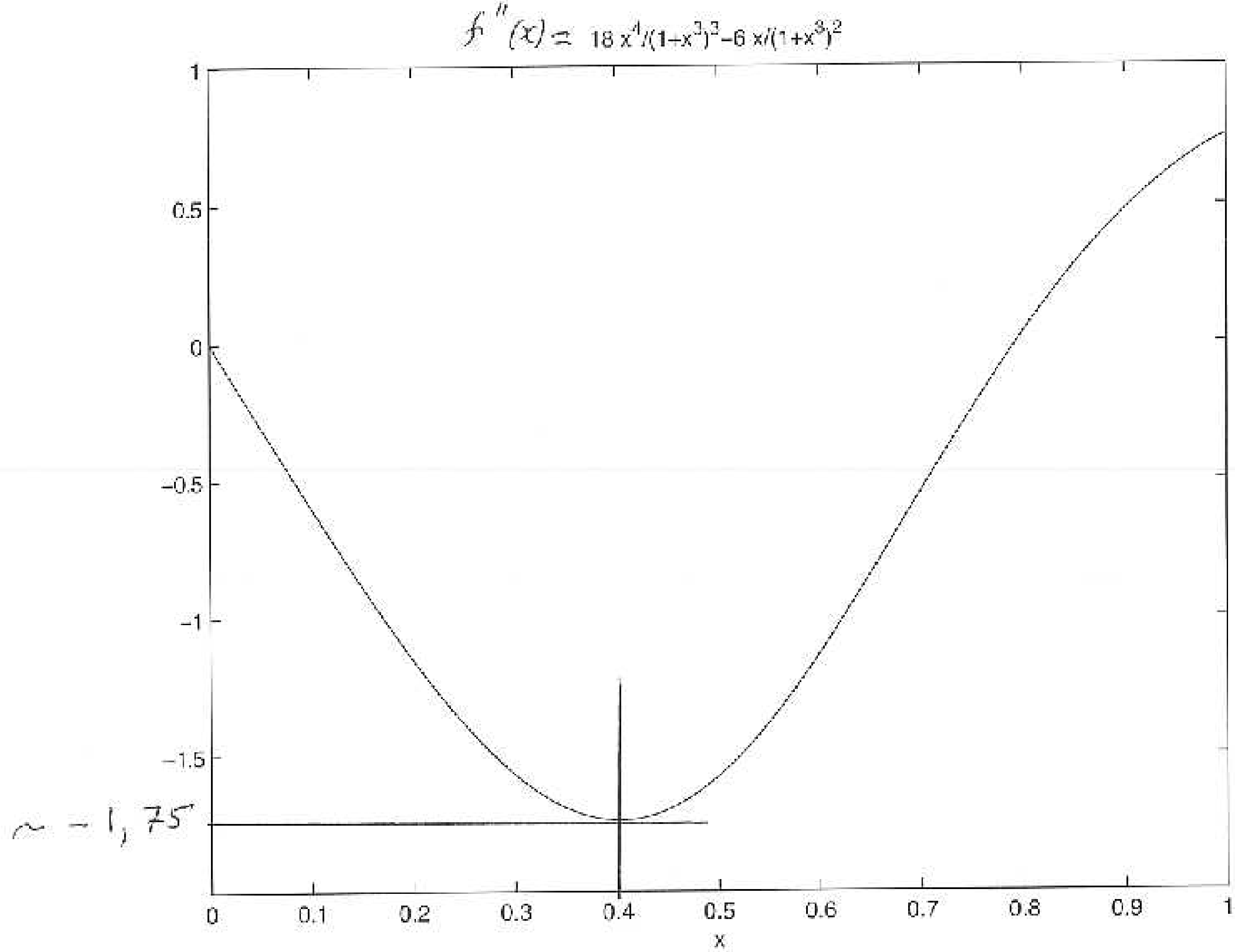
On trouve le max de $|f''(x)|$ au moyen de la solution en x de

$f'''(x) = 0$ sur $0 < x < 1$

Alor $M = \max\{f''(0), f''(1), f''(\bar{x})\}$

Voir le graphique.

DEV. 6 Ex. 10.7



D 6.9