D5.1

MAT 2784A SOLUTIONS

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(1) 3.24 $Ly := y'' - y' - 2y = 2xe^{-x} + x^2$

$\lambda^2 - \lambda - 2 = 0 \qquad (\lambda+1)(\lambda-2) = 0 \qquad \lambda_1 = -1 \quad \lambda_2 = 2$

$y_h(x) = Ae^{-x} + Be^{2x}$

$y_{p_1}(x) = ax^2e^{-x} + bxe^{-x}$

$y_{p_1}'(x) = -ax^2e^{-x} + 2axe^{-x} - bxe^{-x} + be^{-x} = -ax^2e^{-x} + (2a-b)xe^{-x} + be^{-x}$

$y_{p_1}''(x) = ax^2e^{-x} - 4axe^{-x} + 2ae^{-x} + bxe^{-x} - be^{-x} - be^{-x}$
$= ax^2e^{-x} + (b-4a)xe^{-x} + (2a-2b)e^{-x}$

$Ly_{p_1} = -6axe^{-x} + (2a-3b)e^{-x} = 2xe^{-x} + 0e^{-x}$

$-6a = 2 \qquad 2a - 3b = 0$

$a = -\frac{1}{3} \qquad b = -\frac{2}{9}$

$y_{p_2}(x) = ax^2 + bx + c \qquad y_{p_2}'(x) = 2ax + b \qquad y_{p_2}''(x) = 2a$

$Ly_{p_2} = -2ax^2 - (2a+2b)x + (2a - b - 2c)\cdot 1 = x^2 + 0\cdot x + 0\cdot 1$

$-2a = 1 \Rightarrow a = -\frac{1}{2}$

$-2a - 2b = 0 \Rightarrow b = \frac{1}{2}$

$2a - b - 2c = 0 \Rightarrow c = -\frac{3}{4}$

$y(x) = Ae^{-x} + Be^{2x} - \frac{1}{3}x^2e^{-x} - \frac{2}{9}xe^{-x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}$

(2) 3.27 $y^{(4)} - y = 8e^x \qquad y(0)=0 \quad y'(0)=2 \quad y''(0)=4 \quad y'''(0)=6$

$\lambda^4 - 1 = 0 \qquad (\lambda^2-1)(\lambda^2+1)=0$

$\lambda_1 = i \quad \lambda_2 = -i \quad \lambda_3 = 1 \quad \lambda_4 = -1$

$y_h(x) = A\cos x + B\sin x + Ce^x + De^{-x}$

$y_p(x) = axe^x \quad y_p'(x) = axe^x + ae^x \quad y_p''(x) = axe^x + 2ae^x \quad y_p'''(x) = axe^x + 3ae^x$
$y_p^{(4)}(x) = axe^x + 4ae^x$

$Ly_p(x) = 4ae^x = 8e^x$
$a = 2$

$y(x) = A\cos x + B\sin x + Ce^x + De^{-x} + 2xe^x$
$y'(x) = -A\sin x + B\cos x + (C+2)e^x - De^{-x} + 2xe^x$
$y''(x) = -A\cos x - B\sin x + (C+4)e^x + De^{-x} + 2xe^x$
$y'''(x) = A\sin x - B\cos x + (C+6)e^x - De^{-x} + 2xe^x$

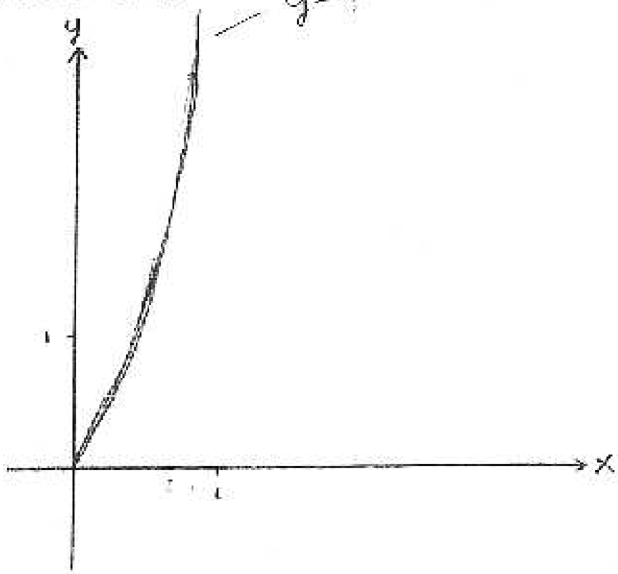
$y(0) \Rightarrow 0 = A + C + D \Rightarrow A = -(C+D)$
$y'(0) \Rightarrow 2 = B + C - D + 2 \Rightarrow B = -(C-D)$
$y''(0) \Rightarrow 4 = -A + C + D + 4 \Rightarrow A = C + D$
$y'''(0) \Rightarrow 6 = -B + C - D + 6 \Rightarrow B = C - D$

$A=0 \quad B=0 \quad C=0 \quad D=0$

Solution unique
$y(x) = 2xe^x$

D5.3

Graphique de 3.27 $\quad y = 2xe^{x} \quad x \geq 0$



(3) 3.31 $\quad y'' + y = \frac{1}{\cos x} = \sec x \qquad \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm i$

$$y_h(x) = A\cos x + B\sin x$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad y_p(x) = c_1(x)\cos x + c_2(x)\sin x$$

$$\underbrace{\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}}_{I} \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}\begin{bmatrix} 0 \\ \sec x \end{bmatrix}$$

matrice orthogonale

$$c_1'(x) = -\tan x$$
$$c_2'(x) = 1$$

$$c_1(x) = -\ln|\cos x| = \ln|\sec x|$$
$$c_2 = x$$

Solution générale

$$y(x) = A\cos x + B\sin x + \ln|\sec x|\cos x + x\sin x$$

(4) 3.35 $y'' + 3y' + 2y = \frac{1}{1+e^x}$

$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow (\lambda+1)(\lambda+2) = 0$

$\lambda_1 = -1 \quad \lambda_2 = -2$

$y_{h}(x) = Ae^{-x} + Be^{-2x}$

$y_p(x) = C_1(x)e^{-x} + C_2(x)e^{-2x}$

$$\begin{bmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{1+e^x} \end{bmatrix}$$

$$\begin{bmatrix} e^{-x} & e^{-2x} \\ 0 & -e^{-2x} \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{1+e^x} \end{bmatrix}$$

$$\begin{bmatrix} e^{-x} & 0 \\ 0 & -e^{-2x} \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{1+e^x} \\ \frac{1}{1+e^x} \end{bmatrix}$$

$e^{-x} C_1'(x) = \frac{1}{1+e^x} \Rightarrow C_1'(x) = \frac{e^x}{1+e^x}$

$-e^{-2x} C_2'(x) = \frac{1}{1+e^x} \Rightarrow C_2'(x) = \frac{-e^{2x}}{1+e^x}$

par substitution $u = 1+e^x$ $\quad C_1(x) = \int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{u}\,du = \ln|u| = \ln(1+e^x)$

par substitution $u = 1+e^x$ $\quad C_2(x) = -\int \frac{e^{2x}}{1+e^x}\,dx = -\int \frac{u-1}{u}\,du = -\int \left(1 - \frac{1}{u}\right)du = -u + \ln|u|$

$= -1 - e^x + \ln(1+e^x)$ $\qquad C_2(x)e^{-2x} = \underbrace{-e^{-2x} - e^{-x}} + \ln(1+e^x)e^{-2x}$

Solution générale

$y(x) = Ae^{-x} + Be^{-2x} + \ln(1+e^x)e^{-x} + \ln(1+e^x)e^{-2x}$

On peut les enlever parce qu'ils sont inclus dans la solution homogène.

D 5.5

(5) 3.38 $2x^2y'' + xy' - 3y = x^{-2}$ $\quad y(1) = 0 \quad y'(1) = 2$

$$2m^2 - m - 3 = 0 \qquad m_1 = -1 \quad m_2 = \frac{3}{2}$$

$$y_h(x) = Ax^{-1} + Bx^{\frac{3}{2}}$$

$$y_p(x) = C_1(x)x^{-1} + C_2(x)x^{\frac{3}{2}}$$

$$\begin{bmatrix} x^{-1} & x^{\frac{3}{2}} \\ -x^{-2} & \frac{3}{2}x^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}x^{-4} \end{bmatrix}$$

$$\begin{bmatrix} x^{-1} & x^{\frac{3}{2}} \\ 0 & \frac{5}{2}x^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}x^{-4} \end{bmatrix}$$

$$x^{-1}C_1'(x) + x^{\frac{3}{2}}C_2'(x) = 0 \implies C_1'(x) = -\frac{1}{5}x^{-2}$$
$$\frac{5}{2}x^{\frac{1}{2}}C_2'(x) = \frac{1}{2}x^{-4} \implies C_2'(x) = \frac{1}{5}x^{-\frac{9}{2}}$$

$$\left.\begin{aligned} C_1(x) &= -\frac{1}{5}\int x^{-2}\,dx = \frac{1}{5}x^{-1} \\ C_2(x) &= \frac{1}{5}\int x^{-\frac{9}{2}}\,dx = -\frac{2}{35}x^{-\frac{7}{2}} \end{aligned}\right\} \quad y_p(x) = \frac{1}{7}x^{-2}$$

$$y(x) = Ax^{-1} + Bx^{\frac{3}{2}} + \frac{1}{7}x^{-2}$$

$$y'(x) = -Ax^{-2} + \frac{3}{2}Bx^{\frac{1}{2}} - \frac{2}{7}x^{-3}$$

$$y(1) \implies 0 = A + B + \frac{1}{7} \implies B = -A - \frac{1}{7}$$
$$y'(1) \implies 2 = -A + \frac{3B}{2} - \frac{2}{7} \implies A = \frac{3}{2}B - \frac{2}{7} - 2$$

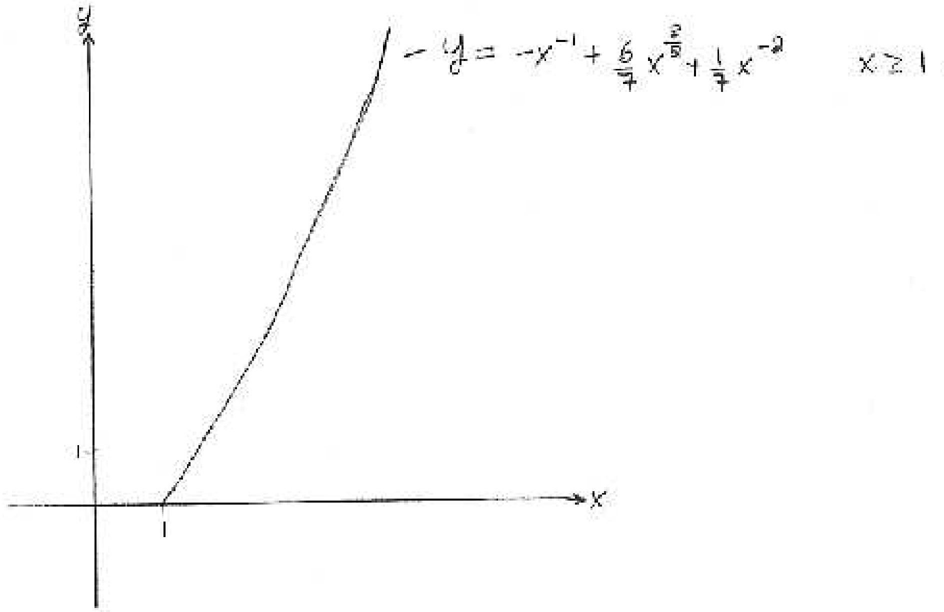
$$A = \frac{3}{2}B - \frac{16}{7} \implies B = -\frac{3}{2}B + \frac{16}{7} - \frac{1}{7}$$

Solution unique

$$B = \frac{6}{7}$$
$$A = -1$$

$$y(x) = -x^{-1} + \frac{6}{7}x^{\frac{3}{2}} + \frac{1}{7}x^{-2}$$

Graphique 3.38



6 4.1 $y' = Ay \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = \lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+2)(\lambda+1) = 0$$

$$\lambda_1 = -1 \quad \lambda_2 = -2$$

$$(A + I)u = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$\left.\begin{array}{l} u_1 + u_2 = 0 \\ -2u_1 - 2u_2 = 0 \end{array}\right\} u_1 = -u_2$$

on pose $u_1 = 1$
alors $u_2 = -1$

$$(A + 2I)v = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\left.\begin{array}{l} 2v_1 + v_2 = 0 \\ -2v_1 - v_2 = 0 \end{array}\right\} v_1 = \frac{-v_2}{2}$$

on pose $v_1 = 1$
alors $v_2 = -2$

<u>Solution générale</u>

$$y(x) = Y(x)c \Rightarrow y(x) = \begin{bmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{bmatrix} c$$

7 9.10 (-1;3) (0;1) (1;0) (2;5)

Polynôme de Gregory-Newton prograde de degré 3

$$P_n(r) = f_0 + \sum_{K=1}^{n} \frac{r(r-1)\cdots(r-K+1)}{K!} \Delta^K f.$$

$$P_3(r) = f_0 + r\Delta f + \frac{r(r-1)}{2}\Delta^2 f + \frac{r(r-1)(r-2)}{6}\Delta^3 f$$

i	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
0	-1	$\boxed{3}$			
			$\boxed{-2}$		
1	0	1		$\boxed{1}$	
			-1		$\boxed{5}$
2	1	0		6	
			5		
3	2	5			

$$P_3(r) = 3 + r(-2) + \frac{r(r-1)}{2}(1) + \frac{r(r-1)(r-2)}{6}(5)$$

$$P_3(r) = 3 - 2r + \frac{1}{2}r(r-1) + \frac{5}{6}r(r-1)(r-2)$$

(8) 9.13 Hermite degré 3

x	$f(x)$	$f'(x)$
8,3	17,56492	3,116256
8,6	18,50515	3,151762

$$P_{2n+1}(x) = \sum_{m=0}^{n} h_m(x) f_m + \sum_{m=0}^{n} \hat{h}_m(x) f'_m$$
$$= f[z_0] + \sum_{k=1}^{2n+1} f[z_0, z_1, \ldots, z_k](x-x_0)(x-z_1)\ldots(x-z_{k-1})$$

z	$f(z)$	1er DD	2e DD	3e DD
8,3	$\boxed{17,56492}$			
		$\boxed{3,116256}$		
8,3	17,56492		$\boxed{0,05948}$	
		3,1341		$\boxed{-2,022 \times 10^{-3}}$
8,6	18,50515		0,0588733	
		3,151762		
8,6	18,50515			

$$P_{2n+1}(x) = 17,56492 + (x-8,3)(3,116256) + (x-8,3)^2(0,05948) - (x-8,3)^2(x-8,6)(0,002022)$$