NOM / NAME SOLUTIONS

Nº d'ÉT. / ST. #

Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

1/8
+p 9

Test mi-session 2
Durée: 80 min
Place: LMX 219
14 novembre 2006
11:30–12:50

Prof.: Rémi Vaillancourt

MAT 2784 A

Midterm 2
Time: 80 min
Place: LMX 219
14 November 2006
11:30–12:50

Instructions:

(a) *À livre fermé. Tout type de calculatrices autorisé.*
Closed book. All types of calculators are allowed.

(b) *Répondre sur le questionnaire. Réponses numériques dans les boîtes.*
Answer on the question sheets. Fill-in boxes with numerical answers.

(c) *Les 7 questions sont d'égale valeur.*
All seven questions have the same value.

(d) *Donner le détail de vos calculs.*
Show all computation.

(e) *Une feuille couleur de tables sera distribuée.*
A one-page table on colored paper will be distributed.

(f) *Tous les angles sont en RADIANS. Tester et ajuster votre calculatrice.*
All angles are in RADIAN measures. Test and adjust your calculators.

$$\sin 1.123456789 = 0.90160112364453$$

1	/10
2	/10
3	/10
4	/10
5	/10
6	/10
7	/10
TOTAL	/70

585, av. King-Edward
Ottawa (Ontario) K1N 6N5 Canada

585 King Edward Avenue
Ottawa, Ontario K1N 6N5 Canada

(613) 562-5864 • Téléc./Fax (613) 562-5776
Courriel/Email: uomaths@science.uottawa.ca

2/8

Qu. 1. *Résoudre le systeme homogène:* / Solve the homogeneous system.:

$$y' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} y.$$

$(A - I\lambda) = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2$

$(\lambda+1)(\lambda+2)$

$\lambda = -1$ $\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $u_1 + u_2 = 0$ posons $u_1 = 1, u_2 = -1$

$\lambda = -2$ $\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $2v_1 + v_2 = 0$ posons $v_1 = 1, v_2 = -2$

$$\boxed{y(x) = c_1 e^{-x}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2x}\begin{bmatrix} 1 \\ -2 \end{bmatrix}}$$

 3/8

Qu. 2. *Résoudre le système nonhomogène :* / Solve the nonhomogeneous system:

$$y' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} y + \begin{bmatrix} 1 \\ x \end{bmatrix}.$$

$$(A - I\lambda) = \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = -4 + \lambda^2 + 3 = \lambda^2 - 1$$

$$(\lambda+1)(\lambda-1)$$

$\lambda = -1$ $\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $3u_1 - u_2 = 0$ posons $u_1 = 1$, $u_2 = 3$

$\lambda = 1$ $\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $v_1 - v_2 = 0$ posons $v_1 = 1$, $v_2 = 1$

$y_h(x) = c_1 e^{-x}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{x}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad y_p'(x) = A y_p(x) + f(x)$

$f(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad y_p(x) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} x$

$y_p'(x) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$b_1 = 2a_1 - a_2 + 1$ $\qquad 0 = 2a_1 - a_2$ $\quad a_2 = 2a_1$

$b_2 = 3a_1 - 2a_2$ $\qquad 2 = 3a_1 - 2a_2$ $\quad 2 = 3a_1 - 4a_1$

$a_1 = -2$ $\quad a_2 = -4$

$0 = 2b_1 - b_2$ $\qquad b_2 = 2b_1$

$0 = 3b_1 - 2b_2 + 1$ $\qquad 0 = 3b_1 - 2(2b_1) + 1$

$b_1 = 1$ $\quad b_2 = 2$

$$\boxed{y(x) = c_1 e^{-x}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{x}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} + x\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

4/18

Qu. 3. *Trouver le rayon de convergence de la série :*

Find the radius of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{4^n}{n^2\, 3^{n+2}}\, x^n.$$

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \qquad \lim_{n\to\infty} \left| \frac{4^{\cancel{n}+1}}{(n+1)^2\, 3^{\cancel{n+3}}} \cdot \frac{n^2 \cdot \cancel{3^{n+2}}}{\cancel{4^n}} \right| = \lim_{n\to\infty} \left| \frac{4}{3} \frac{n^2}{(n+1)^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{4}{3} \frac{\cancel{n^2}}{\cancel{n^2}(1+\underbrace{\tfrac{1}{n}}_{0})^2} \right| = \frac{4}{3}$$

$$\frac{1}{R} = \frac{4}{3} \qquad \boxed{R = \frac{3}{4}}$$

Qu. 4. *Trouver les 6 premiers termes et le terme général de la solution série de l'équation différentielle :*

Find the first six terms and the general term of the series solution to the differential equation:

$$y'' - xy' - y = 0.$$

$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6$

$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + 7a_7 x^6$

$y''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + 56a_8 x^6$

$-y: -a_0 - a_1 x - a_2 x^2 - a_3 x^3 - a_4 x^4 - a_5 x^5 - a_6 x^6$

$-xy': -a_1 x - 2a_2 x^2 - 3a_3 x^3 - 4a_4 x^4 - 5a_5 x^5 - 6a_6 x^6$

$y'': 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + 56a_8 x^6$

$x^0: 0 = 2a_2 - a_0$ $\quad x^1: 0 = -a_1 - a_1 + 6a_3$ $\quad x^2: 0 = -a_2 - 2a_2 + 12a_4$ $\quad x^3: 0 = -a_3 - 3a_3 + 20a_5$

$a_2 = \frac{a_0}{2}$ $\quad a_3 = \frac{a_1}{3}$ $\quad a_4 = \frac{a_2}{4} = \frac{a_0}{8}$ $\quad a_5 = \frac{a_3}{5} = \frac{a_1}{15}$

a_0 et a_1 sont indéterminés

$x^4: a_6 = \frac{a_4}{6} = \frac{a_0}{48}$ $\quad x^5: a_7 = \frac{a_5}{7} = \frac{a_1}{105}$ $\quad x^6: a_8 = \frac{a_6}{8} = \frac{a_0}{384}$

$$\boxed{x^s: a_{s+2} = \frac{a_s}{(s+2)} \qquad x^s: 0 = -a_s - s a_s + (s+1)(s+2) a_{s+2}}$$

6/8

Qu. 5. *Soit* / Let :

$$f(x) = \begin{cases} x^2, & x < 0, \\ x^{1/2}, & x > 0. \end{cases}$$

Calculer les coefficients a_0 et a_1 de Fourier–Legendre pour :

Compute the Fourier–Legendre coefficients a_0 and a_1 for:

$$f(x) = a_0\, P_0(x) + a_1\, P_1(x) + a_2\, P_2(x) + \dots, \qquad -1 \le x \le 1.$$

Noter / Note:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1),$$

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx, \qquad m = 0, 1, 2, \dots$$

$$a_0 = \lim_{t \to 0} \frac{1}{2}\int_{-1}^{t} x^2\, dx + \lim_{t \to 0} \frac{1}{2}\int_{t}^{1} x^{1/2}\, dx$$

$$= \lim_{t \to 0} \frac{1}{2}\left(\frac{x^3}{3}\right)_{-1}^{t} + \lim_{t \to 0} \frac{1}{2}\left(\frac{2}{3}x^{3/2}\right)_{t}^{1}$$

$$= \lim_{t \to 0} \frac{1}{2}\left(\frac{\overset{\to 0}{t^3}}{3} - \frac{(-1)^3}{3}\right) + \lim_{t \to 0} \frac{1}{2}\left(\frac{2}{3} - \frac{2}{3}\overset{\to 0}{t^{3/2}}\right)$$

$$= \frac{1}{6} + \frac{1}{3} = \boxed{\frac{1}{2}}$$

$$a_1 = \lim_{t \to 0} \frac{3}{2}\int_{-1}^{t} x^3\, dx + \lim_{t \to 0} \frac{3}{2}\int_{t}^{1} x^{3/2}\, dx$$

$$= \lim_{t \to 0} \frac{3}{2}\left(\frac{x^4}{4}\right)_{-1}^{t} + \lim_{t \to 0} \frac{3}{2}\left(\frac{2}{5}x^{5/2}\right)_{t}^{1}$$

$$= \lim_{t \to 0} \frac{3}{2}\left(\frac{\overset{\to 0}{t^4}}{4} - \frac{1}{4}\right) + \lim_{t \to 0} \frac{3}{2}\left(\frac{2}{5} - \frac{2}{5}\overset{\to 0}{t^{5/2}}\right)$$

$$= \frac{-3}{8} + \frac{3}{5} = \frac{-15}{40} + \frac{24}{40} = \boxed{\frac{9}{40}}$$

Qu. 6. *Soit* / Given :

$$I = \int_{-1}^{1} e^{-x^2} dx \approx 1.494.$$

Évaluer I, au millonième près, par la quadrature gaussienne à 3 points en justifiant votre réponse par les résultats intermédiaires.

Evaluate I, to six decimals, by the three-point Gaussian formula and list intermediary results to justify your answer.

Quadrature gaussienne / Gaussian quadrature :

$$\int_{-1}^{1} f(x)\,dx \sim \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right).$$

$f(x) = e^{-x^2}$

$f(-\sqrt{\tfrac{3}{5}}) = e^{-3/5} = 0{,}5488116$

$f(0) = e^0 = 1{,}000000$

$f(\sqrt{\tfrac{3}{5}}) = e^{-3/5} = 0{,}5488116$

$$\int_{-1}^{1} e^{-x^2} dx \approx \frac{5}{9}(0{,}5488116) + \frac{8}{9} + \frac{5}{9}(0{,}5488116) \approx 1{,}498679$$

$$\approx 0{,}304895 + 0{,}\overline{88} + 0{,}304895 \approx \boxed{1{,}4986795}$$

8

8/8

Qu. 7. *Soit le terme de l'erreur de la méthode des trapèzes pour* $\int_a^b f(x)\,dx$:

The truncation error of the composite trapezoidal rule for $\int_a^b f(x)\,dx$ is:

$$-\frac{(b-a)h^2}{12}f''(\xi).$$

Déterminer les valeurs de h *et* n *pour approcher l'intégrale suivante à* 10^{-4} *près par la méthode des trapèzes :*

Determiner the values of h and n to approximate the following integral to 10^{-4} by the composite trapezoidal rule :

$$\int_0^1 \frac{dx}{1+x^3}$$

sol. p. 9

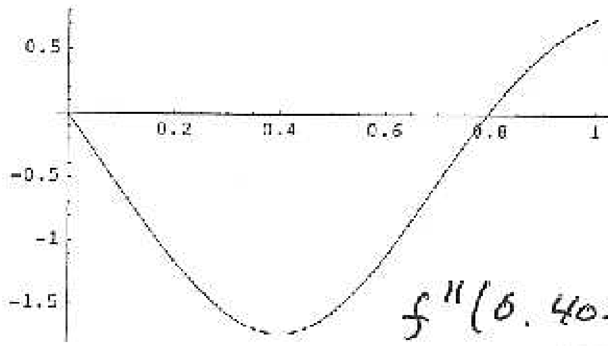
Qu. 7

```
In[2]:= f[x_] := 1 / (1 + x^3)

        Plot[f''[x], {x, 0, 1}]
```

$f''(0) = 0$

$f''(1) = 3/4$

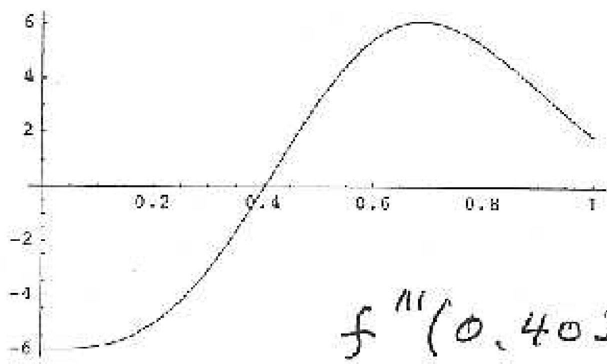


$f''(0.402) \simeq -1.73751$

```
Out[3]= - Graphics -
```

$$\boxed{M = \max_{0 \le x \le 1} |f''(x)| \simeq 1.74}$$

```
In[4]:= Plot[f'''[x], {x, 0, 1}]
```



$f'''(0.402) \simeq 0$

```
Out[4]= - Graphics -
```

$$\frac{1-0}{12} h^2 M < 10^{-4} \Rightarrow h_{max}^2 = \frac{10^{-4} \times 12}{1.74}$$

$$h \simeq 0.026801$$

$$\frac{1}{h_{max}} \simeq 38.0510 \leq 39 = n \text{ un entier.}$$

$$b - a = nh \Rightarrow \frac{1}{h} = \frac{1}{39}$$