MAT 2784
Devoir 7
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\# 3,31 $\quad y'' + y = \dfrac{1}{\cos x}$

Solution homogène: $\quad y'' + y = 0$

Posons $y = e^{\lambda x} \Rightarrow y' = \lambda e^{\lambda x} \Rightarrow y'' = \lambda^2 e^{\lambda x}$

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0$$
$$\lambda^2 + 1 = 0$$
$$\lambda = \pm\sqrt{-1}$$
$$= \pm i$$
$$\lambda = \alpha \pm i\beta.$$

$$y_h = a\cos x + b\sin x$$

Solution particulière:

$$y_p = c_1\cos x + c_2 \sin x$$

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\cos x \end{bmatrix}$$

$$c_1'\cos x + c_2'\sin x = 0$$
$$-c_1'\sin x + c_2'\cos x = \frac{1}{\cos x}$$

$$c_1' = -c_2'\frac{\sin x}{\cos x}$$

$$-\left(-c_2'\frac{\sin x}{\cos x}\right)\sin x + c_2'\cos x = \frac{1}{\cos x}$$

$$c_2' \left(\frac{\sin^2 x}{\cos x} + \cos x \right) = \frac{1}{\cos x}$$

$$c_2' = \frac{1}{\cos x}\left(\frac{\cos x}{\sin^2 x}\right) + \frac{1}{\cos x}\left(\frac{1}{\cos x}\right)$$

$$c_2' = \frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{1}{\sin^2 x + \cos^2 x} = 1$$

$$c_1' = -1\left(\frac{\sin x}{\cos x}\right)$$

$$c_1' = -\operatorname{tg} x$$

$$\Longrightarrow \quad c_2 = \int c_2'\,dx = \int dx = x$$

$$c_1 = \int c_1'\,dx = -\int \operatorname{tg} x\,dx = -\ln|\sec x|$$

donc $\quad y_p = -\ln|\sec x|\cos x + x\sin x$

$$y(x) = y_h + y_p$$

$$\boxed{y(x) = a\cos x + b\sin x - \ln|\sec x|\cos x + x\sin x}$$

\# 6,3 $\quad f(t) = \cos(\omega t + \theta)$

$$f(t) = \cos\omega t\cos\theta - \sin\omega t\sin\theta$$

$$\mathcal{L}(f(t))(s) = \mathcal{L}(\cos\omega t)\cdot\cos\theta - \mathcal{L}(\sin\omega t)\sin\theta$$

$$= \left(\frac{s}{s^2+\omega^2}\right)\cos\theta - \left(\frac{\omega}{s^2+\omega^2}\right)\sin\theta$$

$$\boxed{\mathcal{L}(f(t))(s) = \frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}}$$

#6.3) $f(t) = \cos(\omega t + \theta)$

$= \cos(\omega t) \cdot \cos(\theta) + \sin(\omega t) \cdot \sin(\theta)$

$\mathcal{L}(\cos(\omega t)) \cdot \cos\theta - \mathcal{L}(\sin(\omega t)) \cdot \sin\theta$

$= \left(\frac{s}{s^2+\omega^2}\right)\cos\theta - \left(\frac{\omega}{s^2+\omega^2}\right)\sin\theta$

$$\boxed{= \frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}}$$

#6.7) $f(t) = 3\cosh(2t) + 4\sinh(5t)$

$3\mathcal{L}(\cosh(2t)) + 4\mathcal{L}(\sinh(5t))$

$3 \cdot \left(\frac{s}{s^2-2^2}\right) + 4\left(\frac{5}{s^2-5^2}\right)$

$$\boxed{\frac{3s}{s^2-4} + \frac{20}{s^2-25}}$$

$$\frac{3s}{(s-2)(s+2)} + \frac{20}{(s-5)(s+5)}$$

$$\frac{3s(s-5)(s+5) + 20(s-2)(s+2)}{(s-2)(s+2)(s-5)(s+5)}$$

$$= \frac{3s^3 - 75s + 20s^2 - 80}{s^4 - 4s^2 - 25s^2 + 100}$$

$$= \frac{3s^3 + 20s^2 - 75s - 80}{s^4 - 29s^2 + 100}$$

#6,8 $\quad f(t) = 2e^{-2t}\sin t$

$$\mathcal{L}(f(t))(s) = \int_0^\infty 2e^{-2t}\sin t \cdot e^{-st}\,dt$$
$$= 2\int_0^\infty e^{-st}e^{-2t}\sin t\,dt$$
$$= 2\mathcal{L}(e^{-2t}\sin t)(s)$$

$$\boxed{\mathcal{L}(f(t))(s) = \frac{2}{(s+2)^2+1}}$$

car on sait que

$$\mathcal{L}(e^{at}\sin\omega t) = \frac{\omega}{(s-a)^2+\omega^2}$$

Dans ce cas,
$a=-2$ et $\omega = 1$.

#6.41 $\quad y'' - 6y' + 13y = 0$

$y(0)=0$
$y'(0)=-3$

$$\mathcal{L}(y)(s) = Y(s)$$

$$0 = \mathcal{L}(y'') - 6\mathcal{L}(y') + 13\mathcal{L}(y) = s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 13Y(s)$$

$$(s^2-6s+13)Y(s) - (s-6)y(0) - y'(0) = 0$$
$$(s^2-6s+13)Y(s) = \underbrace{(s-6)y(0)}_{\to 0} + y'(0)$$

$$(s^2-6s+13)Y(s) = -3$$
$$Y(s) = \frac{-3}{s^2-6s+13}$$
$$= -3\left(\frac{1}{s^2-6s+9+4}\right)$$

07.5

$$Y(s) = -3\left(\frac{1}{(s-3)^2+4}\right)$$

$$= -3\left(\frac{1}{(s-3)^2+2^2}\right)$$

$$Y(s) = -3\left(\frac{1}{(s-a)^2+\omega^2}\right)$$

où $a=3$
$\omega=2$

donc $y(t) = -3\left(\frac{1}{\omega}e^{at}\sin\omega t\right)$

$$\boxed{y(t) = \frac{-3}{2}e^{3t}\sin 2t}$$

#6.46 $y'' + 2y' + 5y = 4t$

$y(0)=0$
$y'(0)=0$

$$\mathcal{L}(4t) = \frac{4}{s^2}$$

$$\frac{4}{s} = \mathcal{L}(y'') + 2\mathcal{L}(y') + 5\mathcal{L}(y) = s^2Y(s) - \cancel{sy(0)}^{0} - \cancel{y'(0)}^{0} + 2[sY(s) - \cancel{y(0)}^{0}] + 5Y(s)$$

$$\frac{4}{s^2} = s^2Y(s) + 2sY(s) + 5Y(s)$$

$$\frac{4}{s^2} = (s^2+2s+5)Y(s)$$

$$Y(s) = \frac{4}{s^2(s^2+2s+5)}$$

$$= \frac{4}{s^2(s^2+2s+1+4)}$$

$$Y(s) = \frac{4}{s^2((s+1)^2+4)}$$

$$Y(s) = \frac{A+Bs}{s^2} + \frac{C+Ds}{s^2+2s+5}$$

$$4 = As^2 + 2sA + 5A + s^3B + 2s^2B + 5sB + s^2C + s^3D$$

$$4 = s^3(B+D) + s^2(A+2B+C) + s(2A+5B) + 5A$$

$$5A = 4 \Rightarrow A = 4/5$$

$$2A + 5B = 0 \Rightarrow B = \left(\frac{-8}{5}\right)\frac{1}{5} = -8/25$$

$$B + D = 0 \Rightarrow D = -B = 8/25$$

$$A + 2B + C = 0 \Rightarrow C = -A - 2B = \frac{-4}{5} + \frac{16}{25} = \frac{-4}{25}$$

$$Y(s) = \frac{4/5 - 8s/25}{s^2} + \frac{-4/25 + 8s/25}{s^2+2s+5}$$

$$= \frac{4}{5s^2} - \frac{8}{25s} - \frac{4}{25((s+1)^2+4)} + \frac{8s}{25((s+1)^2+4)}$$

$$= \frac{4}{5s^2} - \frac{8}{25s} - \frac{4}{25((s+1)^2+4)} + \frac{8(s+1-1)}{25((s+1)^2+4)}$$

$$= \frac{4}{5s^2} - \frac{8}{25s} - \frac{4}{25((s+1)^2+4)} + \frac{8(s+1)}{25((s+1)^2+4)} - \frac{8}{25((s+1)^2+4)}$$

$$y(t) = \frac{4}{5}t + -\frac{8}{25} - \frac{4}{25}\cdot\frac{1}{2}e^{-t}\sin 2t + \frac{8}{25}e^{-t}\cos 2t - \frac{8}{25}\cdot\frac{1}{2}e^{-t}\sin 2t$$

$$\boxed{y(t) = \frac{-6}{25}e^{-t}\sin 2t + \frac{8}{25}e^{-t}\cos 2t + \frac{4t}{5} - \frac{8}{25}}$$

10.9

$\int_1^3 \ln x \, dx$ à 10^{-3} près

$a = 1$
$b = 3$

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$ et $f''(x) = \frac{-1}{x^2}$

$M = \max_{1 \leq x \leq 3} |f''(x)| = |f''(1)| = \left|\frac{-1}{1^2}\right| = 1$

car, en valeur absolue, $f''(x)$ est décroissan sur l'intervalle de 1 à 3 !

$|f''(1)| = 1$ et $|f''(3)| = \frac{1}{9} = 0,111$

$|f''(1)| > |f''(3)|$

méthode des trapèzes:

$$\left|\frac{(b-a)h^2}{12} f''(\xi)\right| \leq \frac{2}{12} h^2 M = \frac{h^2}{6} \cdot 1$$

$$\frac{h^2}{6} \leq 10^{-3}$$
$$h^2 \leq 6 \cdot 10^{-3}$$
$$h \leq \sqrt{6 \cdot 10^{-3}}$$
$$h \leq 0,07746$$

$\frac{2}{h} = 25,82 \leq n = 26$ $\boxed{n = 26}$

$h = \frac{2}{n} = \frac{2}{26} = 0,0769$ $\boxed{h = 0,769}$

méthode des points milieux:

$$\left| \frac{(b-a)h^2}{24} f''(\xi) \right| \leq \frac{2}{24} h^2 \cdot M = \frac{h^2}{12} \cdot 1$$ p7. 8

$$\frac{h^2}{12} \leq 10^{-3}$$

$$h^2 \leq 10^{-3} \cdot 12$$

$$h \leq \sqrt{12 \cdot 10^{-3}}$$

$$h \leq 0{,}1095$$

$$\frac{2}{h} = 18{,}26 \leq n = 19 \qquad \boxed{n = 19}$$

$$h = \frac{2}{n} = \frac{2}{19} = 0{,}105 \qquad \boxed{h = 0{,}105}$$

méthode de Simpson:

$$f'''(x) = \frac{2}{x^3} \qquad f^{(4)}(x) = \frac{-6}{x^4}$$

$$M = \max_{1 \leq x \leq 3} |f^{(4)}(x)| = |f^{(4)}(1)| = 6$$

$$\left| \frac{(b-a)h^4}{180} f^{(4)}(\xi) \right| \leq \frac{2}{180} h^4 M = \frac{h^4}{90} \cdot 6$$

$$0{,}0667\, h^4 \leq 10^{-3}$$

$$h^4 \leq 0{,}015$$

$$h \leq \sqrt[4]{0{,}015}$$

$$h \leq 0{,}34996$$

$$\frac{2}{h} = 5{,}72 \leq n = 6 \qquad \boxed{n = 6}$$

$$h = \frac{2}{n} = \frac{2}{6} = 0{,}333 \qquad \boxed{h = 0{,}333}$$

8. $\int_0^2 \frac{1}{x+4}\,dx \rightarrow$ à 10^{-5} près
(10/10)

Trapèzes:

$f(x) = \frac{1}{x+4}$, $f'(x) = -(x+4)^{-2}$, $f''(x) = 2(x+4)^{-3}$
$= \frac{2}{(x+4)^3}$

$\frac{1}{108} \leq f''(x) \leq \frac{1}{32}$ pour $x \in [0,2]$

$|E_T| \leq \left(\frac{1}{32}\right)\left(\frac{1}{12}\right)(2-0)\,h^2 = \frac{h^2}{192} < 10^{-5}$

$h < 0.043817805$

$n \geq \frac{2}{h} = 45.6435\ldots$

$\therefore n = 46$ et $h = \frac{1}{23}$

(page suivante)

Point milieu:

$f''(x) = \frac{-1}{x^2}$

$-1 \leq f''(x) \leq -\frac{1}{9}$ pour $x \in [1,3]$

$|E_T| \leq (1)\left(\frac{1}{24}\right)(3-1)h^2 = \frac{h^2}{12} < 10^{-3}$

$h < 0.109544512$

$n \geq \frac{2}{h} = 18.2574...$

$\therefore n = 19$ et $h = \frac{2}{19}$

8. $\int_0^2 \frac{1}{x+4}\,dx \rightarrow$ à 10^{-5} près
(10.10)

Trapèzes:

$f(x) = \frac{1}{x+4}$, $f'(x) = -(x+4)^{-2}$, $f''(x) = 2(x+4)^{-3} = \frac{2}{(x+4)^3}$

$\frac{1}{108} \leq f''(x) \leq \frac{1}{32}$ pour $x \in [0,2]$

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$\therefore n = 46$ et $h = \frac{1}{23}$

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